Tree Volume and Upper-stem Diameter Predictions for Planted Loblolly and Slash Pine Based on a Compatible Volume-Taper System with Segmented-stem Form Factors

Plantation Management Research Cooperative

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PMRC Technical Report 1999-3

February 1999

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Summary

By integrating a variable-form differential equation for taper, a segmented-stem system of models involving a taper equation, merchantable volume equation, and total stem volume equation results. The model treats the tree stem as having three segments, each with its own form, joined at two inflection points. Through the constraints imposed on the coefficients of integration, the taper model is continuous at the two inflection points and all components in the system are compatible with each other. The constant-form-factor model (no inflection point) and the one inflection point segmented model are special cases of this more general model. All parameters are shared by the components in our system. The fitted system of models is based on PMRC stem analysis data for 1280 loblolly pine (*Pinus taeda* L.) trees and 871 slash pine (*Pinus elliottii* Engelm.) trees. We used a simultaneous parameter estimation technique so that all appropriate information was obtained from the data. For both species, our system improved predictions of stem profile diameters when compared to the constant form factor model. Relative form factors implied by our fitted model are strikingly close to the usual mensurational assumptions of neiloid, paraboloid, and conoid shapes for the lower, middle, and upper portions of the tree stem.

Introduction

Based on a model presented by Bailey (1994), Fang and Bailey (1999) developed a compatible taper model system based on an exponential tip volume assumption and included a constraint that assures the volumes obtained by integrating the taper function are equal to those obtained with a volume equation. Their system is flexible in that four cases arise from the same basic model assumption. The total volume equation acts as a constraint on the taper model in each case. The exponential tip volume equation is reasonable in most cases. Coefficients and the independent variables dbh and total tree height at an averaged level for Cases I-b, II-a and II-b in the Fang and Bailey (1999) system can reflect inflection points along a bole. However, the assumption of constant form (b) in their tip volume equation is questionable in some situations and is actually one of the main sources of error in their fitted taper models. For example, when b is constant the simplest taper model (Case I-a, Fang and Bailey 1999) can not express any inflection point on the bole. This property limits the model's use in some species that have obvious inflection points on the bole. One correction for this shortcoming is to generalize the b parameter so that a variable form taper model can be developed.

Foresters have a long history of being aware of the variability in individual tree form and trying to model it so that the overall stem profile can be described more precisely (Behre 1923, 1927). At least two approaches are used with success today. One expresses variable form as a single continuous function. The other expresses variable form with a step function in such a way that the bole is separated into segments by inflection points with form being constant within each segment and different between segments. Examples of the first approach are the variable-exponent taper models by Newnham (1988,1992) and Kozak (1988). In these a single continuous function with a changing exponent from ground to top describes the neiloid, paraboloid, and several intermediate forms (Kozak 1997). A separated-exponents bole model by Ormerod (1973) , a segmented polynomial taper model by Max and Burkhart(1976), Goulding and Murray (1976), and Cao et al. (1980) and a conditional dual-equation taper model by Demaerschalk and Kozak (1977) are examples of the second approach. The first approach has some advantages because the shape of a bole is described by a single continuous function and upper-stem diameter predictions are improved as compared to constant-form taper functions. Additionally, the variable-exponent taper model can be transformed so that linear regression techniques are readily applied. But, the terms in the model are usually chosen empirically rather than geometrically and stem volume can not necessarily be obtained by directly integrating the taper function. In such situations, compatibility between taper and volume is difficult to establish.

The second approach noted above simply uses different models for different parts of the stem, thus each can be reasonably simple. Direct integration of the taper model and compatibility between taper and volume are more readily established with this approach (McClure and Czaplewski 1986, Byrne and Reed, 1986). Since the joint points are usually estimated in the process of fitting the model to data, nonlinear techniques are required in parameter estimation.

In this paper, we use the second approach to develop a segmented-stem taper model based on a variable-form differential equation presented by Fang and Bailey (1999). A compatible model for merchantable and total volume is obtained by direct integration of the taper model. We fitted the equation system, which includes a taper equation, a merchantable volume equation, and a total volume equation, with data for both loblolly pine (*Pinus taeda* L.) and slash pine (*Pinus elliottii* Engelm.) using parameter estimation techniques developed for systems of simultaneous equations.

Model derivation

Fang and Bailey (1999) derived a differential equation for taper based on the exponential tip volume equation presented by Bailey (1994). Their model is:

$$\frac{\partial b(.)}{\partial h}+\frac{q_1 b(.)}{d}\frac{\partial d}{\partial h}=-kd_i^{\,2-q_1}(H-h)^{-q_2}+\frac{q_2 b(.)}{H-h} \qquad , \qquad (1)$$

where

β(.) and θ_1, θ_2 are parameters in the Bailey (1994) tip volume equation,

$$\mathrm{V_{tip}} = b(.)\,\mathrm{d}^{q_1}(H-h)^{q_2} \qquad , \qquad (2)$$

and b(.) may change on a stem from ground to the top. b(.) can be a function of h , the height ratio h/H, D alone, or H alone. The variables are:

D = diameter at breast height (cm),
H = total tree height (m),
d = upper-stem diameter (cm) at height h,
h = the length (m) from ground to upper-stem diameter d, and
k = the metric constant π/40000 to convert from diameter in cm to cross-section area in m^2.

Failing in attempts to obtain consistent differential equations for b(.) and d with respect to h, Fang and Bailey (1999) presented equation (1) without further development. Although it is not possible to obtain one explicit function to express continuous form change with (1), we may analyze (1) in segments. We assume the form factor of segment i on a bole is b_i, with b_i constant in segment i but possibly changing between segments. So, in segment i we have $\partial b(.)/\partial h = \partial b_i/\partial h = 0$. equation (1) may then be written

$$\frac{\partial d}{\partial h}-\frac{q_2}{q_1}\frac{d}{H-h}=-\frac{k}{q_1 b_i}\,d^{\,3-q_1}\left(H-h\right)^{-q_2} \qquad , \qquad (3)$$

which is the constant-form equation investigated thoroughly by Fang and Bailey(1999). They present the solutions to (3) in four cases according to the specified values of parameters q_1 and q_2, with the case defined by q_1=2 and q_2 =1 being the simplest one. Since other general cases have not improved model performance for many tree species (Fang and Bailey 1999, Newberry et al. 1989), we decided to use this simplest case in our work. The solution, as presented by Fang and Bailey (1999), is

$$d=c_i\left(H-h\right)^{(k-b_i)/2b_i} \qquad , \qquad (4)$$

where c_i and b_i are the integration constant and form factor of section i, respectively.

The usual diagram for stem taper shows discrete segments of a main stem of a tree separated by inflection points and having specific models for taper, such as neiloid, paraboloid, and cone, within each segment (see Avery and Burkhart 1994, p. 54). The number of inflection points may, in reality, depend on species, stand conditions, and possibly other factors. Two inflection points may well be adequate for most species; one close to dbh and another in an upper position on the bole. Although there is no mathematical necessity, the following derivations are based on the assumption that two inflection points (i.e., three segments) will be sufficient. So, our results have i=1, 2, 3 in equation (4), denoting the three sections from ground to the top separated by the two inflection points.

Suppose the two inflection points occur at height h_1 and h_2 (both from ground) so that the proportions of these heights to total tree height are p_1 and p_2 respectively:

$$p_1 = h_1/H, \qquad p_2 = h_2/H$$

Then model (4) can be specified as:

$$d = c_1 (H-h)^{(k-b_1)/2b_1} \quad , \text{ if } 0 \le h_i/H \le p_1,$$
$$= c_2 (H-h)^{(k-b_2)/2b_2} \quad , \text{ if } p_1 < h_i/H \le p_2,$$
$$= c_3 (H-h)^{(k-b_3)/2b_3} \quad , \text{ if } p_2 < h_i/H \le 1 \ .$$

We determined the constants of integration, c_i , in section i (i=1,2,3) based on the following constraints:

1. Continuity constraints: The taper should be continuous at inflection points h_1 and h_2. Namely,

$$c_1(H-h_1)^{(k-b_1)/2b_1} = c_2(H-h_1)^{(k-b_2)/2b_2}$$

and

$$c_2(H-h_2)^{(k-b_2)/2b_2} = c_3(H-h_2)^{(k-b_3)/2b_3} \quad .$$

The Continuity constraints imply the following relationships:

$$c_2 = c_1(H-h_1)^{(b_2-b_1)k/2b_1b_2}$$
$$c_3 = c_2(H-h_2)^{(b_3-b_2)k/2b_2b_3}$$

Expressed in term of p_1, p_2 and c_1 we have:

$$c_2 = c_1[H(1-p_1)]^{(b_2-b_1)k/2b_1b_2} \quad (5)$$
$$c_3 = c_1 H^{(b_3-b_1)k/2b_1b_3}(1-p_1)^{(b_2-b_1)k/2b_1b_2}(1-p_2)^{(b_3-b_2)k/2b_2b_3} \quad . \quad (6)$$

2. Compatibility constraints: Total volumes obtained by integrating the taper function should be equal to those estimated by a total volume equation,

$$\int_{h_0}^{H} k\, d^2\, dh = V(D,H) \quad .$$

For the segmented model

$$k\left(\int_{h_0}^{h_1} d^2\, dh + \int_{h_1}^{h_2} d^2\, dh + \int_{h_2}^{H} d^2\, dh\right) = V(D,H) ,$$

where h_0 is the height of stump and $V(D,H)$ is a total volume equation as a function of dbh (D) and total height (H). From equation (4), we can obtain

$$\int k\, d^2\, dh = -c_i^2\, b_i (H-h)^{k/b_i}$$

for section i. Hence,

$$-c_1^2 b_1 (H-h)^{k/b_1}\Big|_{p_0H}^{p_1H} + [-c_2^2 b_2 (H-h)^{k/b_2}]\Big|_{p_1H}^{p_2H} + [-c_3^2 b_3 (H-h)^{k/b_3}]\Big|_{p_2H}^{H} = V(D,H) .$$

Substitution of (5) and (6) for c_2 and c_3 into the above equation results in:

$$c_1 = \sqrt{V(D,H) H^{-k/b_1} / [b_1(t_0 - t_1) + b_2(t_1 - a_1 t_2) + b_3 a_1 t_2]} \quad (7)$$

where

$$t_0 = (1-p_0)^{k/b_1} \quad , \quad (8)$$
$$t_1 = (1-p_1)^{k/b_1} \quad , \quad (9)$$
$$t_2 = (1-p_2)^{k/b_2} \quad , \quad (10)$$
$$a_1 = (1-p_1)^{(b_2-b_1)k/b_1b_2} \quad , \quad (11)$$

and $p_0 = h_0/H$.

Letting $z = h / H$, the taper models for the segments are:

$$d = c_1 \sqrt{H^{(k-b_1)/b_1}(1-z)^{(k-b_1)/b_1}} \quad , \text{ if } 0 \le z \le p_1 , \quad (12a)$$

and $$d = c_2 \sqrt{H^{(k-b_2)/b_2} (1-z)^{(k-b_2)/b_2}}\text{, if } p_1 < z \le p_2 \quad . \qquad (12b)$$

Substituting (5) into (12b) gives

$$d = c_1 \sqrt{a_1 H^{(k-b_1)/b_1} (1-z)^{(k-b_2)/b_2}}\text{ , if } p_1 < z \le p_2 \quad . \qquad (13)$$

Similarly,

$$d = c_1 \sqrt{a_1 a_2 H^{(k-b_1)/b_1} (1-z)^{(k-b_3)/b_3}}\text{ , if } p_2 < z \le 1, \qquad (14)$$

where

$$a_2 = (1-p_2)^{(b_3-b_2)k/b_2 b_3} \quad . \qquad (15)$$

We may write the complete system [i.e. (12a), (13) and (14)] as one equation by introducing two dummy variables, I_1 and I_2 , where

$$I_1 = \begin{cases} 1 & \textit{If} \quad p_1 \le z \le p_2 \\ 0 & \textit{Otherwise} \end{cases} \quad \text{and}$$

$$I_2 = \begin{cases} 1 & \textit{If} \quad p_2 < z \le 1 \\ 0 & \textit{Otherwise} \end{cases} \quad .$$

The segmented-stem model becomes

$$d = c_1 \sqrt{H^{(k-b_1)/b_1} (1-z)^{(k-b)/b} a_1^{I_1+I_2} a_2^{I_2}} \qquad (16)$$

where $$b = b_1^{1-(I_1+I_2)} b_2^{I_1} b_3^{I_2} \quad . \qquad (17)$$

Equation (16) is a continuous function describing the taper for all three segments. For example, when $0 \le z \le p_1$ both I_1 and I_2 are 0, $b = b_1$, and (16) is reduced to (12a). If $p_1 < z \le p_2$, then $I_1 = 1$, $I_2 = 0$, and $b = b_2$ giving rise to (13). If $p_2 < z \le 1$ then $I_1 = 0$, $I_2 = 1$, and $b = b_3$ resulting in (14).

If $h = H$, $z = 1$, then $d = 0$, namely, d is 0 at the top. Moreover, if $b_1 = b_2 = b_3$, then $c_1 = c_2 = c_3$, $\alpha_1 = \alpha_2 = 1$, and (16) is reduced to

$$d = c_1 \sqrt{(H-h)^{(k-b)/b}} \text{ ,}$$

with equation (7) simplified to

$$c_1 = \sqrt{V(D,H) H^{-k/b} / b\, t_0} \quad .$$

This simplified version is Case I-a in Fang and Bailey (1999) with

$$V(D,H) = a_0 D^{a_1} H^{a_2}$$

and stump height =0 (so $p_0 = 0$). Therefore, Fang and Bailey's constant-b taper model (Case I-a) is a special case of the above segmented-stem taper model.

Another case to consider is when $b_1 = b_2$, or $b_2 = b_3$. In this case, $p_1 = p_2$, $I_1 = 0$, $\alpha_1 = 1$ (or $\alpha_2 = 1$), and the segmented-stem taper model with two inflection points will be reduced to a segmented-stem model with only one inflection point. The corresponding adjustment of equation (16) is obvious. Therefore, a segmented-stem model with two segments is a special case of equation (16).

A merchantable volume model can be obtained by integrating the taper model, but integration should done with caution since the form factors may vary from segment to segment. For the three segments we have:

$$V_m = c_1^2 H^{k/b_1} b_1 [t_0 - (1-z)^{k/b_1}] \text{ , if } 0 \le z \le p_1 \text{ ,} \qquad (18)$$

$$= c_1^2 H^{k/b_1} \left(b_1 (t_0 - t_1) + b_2 [t_1 - a_1 (1-z)^{k/b_2}] \right) \text{ , if } p_1 < z \le p_2, \qquad (19)$$

$$= c_1^2 H^{k/b_1} \left(b_1 (t_0 - t_1) + b_2 (t_1 - a_1 t_2) + b_3 a_1 [t_2 - a_2 (1-z)^{k/b_3}] \right)\text{, if } p_2 < z \le 1, \qquad (20)$$

where V_m is the merchantable volume (m^3) and all other terms are defined above.

The merchantable volume models for the three sections can also be rearanged into one equation:

$$V_m = c_1^{\ 2} H^{k/b_1}\left(b_1 t_0 + (I_1 + I_2)(b_2 - b_1)t_1 + I_2(b_3 - b_2)a_1 t_2 - b(1-z)^{k/b} a_1^{\ I_1+I_2} a_2^{\ I_2}\right) \tag{21}$$

As before, logical special cases result. When $0 \le z \le p_1$, both I_1 and I_2 are 0, $b = b_1$, and (21) is reduced to (18). When $p_1 < z \le p_2$ then $I_1 = 1$, $I_2 = 0$, $b = b_2$, and (21) is reduced to (19). When $p_2 < z \le 1$ then $I_1 = 0$, $I_2 = 1$, $b = b_3$, and (21) is reduced to (20). The following compatibility boundary conditions are automatically satisfied by the segmented-stem merchantable volume model: ① When merchantable height equals stump height ($z = p_0$), merchantable volume=0; ② When merchantable height equals total tree height (z=1), merchantable volume equals total tree volume. When all three segments have the same form factor, $b_1 = b_2 = b_3 = b$, then

$$V_m = V(D,H)\left(1 - \left(\frac{1-z}{1-p_0}\right)^{k/b}\right) \tag{22}$$

Equation (22) is the merchantable volume model with a constant form factor referred to by Bailey (1994) and Newberry et al. (1989).

Model Fitting

Data description

The models derived above were fitted to both loblolly pine (*Pinus taeda* L.) and slash pine (*Pinus elliottii* Engelm.) data sets. The loblolly pine data consist of measurements on 1280 individual trees obtained from 376 sample plots located in cutover plantations in the coastal plain and piedmont physiographic provinces of North Carolina, South Carolina, Georgia, Florida and Alabama. The slash pine data consist of measurements on 871 individual trees obtained from 256 sample plots located in cutover plantations in the coastal plain of Georgia and north Florida. In most cases, four sample trees without any obvious stem abnormalities were felled on each sample plot. Two trees classified as dominant or codominant were selected from the larger dbh classes, a third tree was selected from the average dbh class, and a fourth tree was selected from a dbh class smaller than the average dbh class from trees classified as intermediate or suppressed. Each sample tree was measured for dbh with a diameter tape and, after felling, for total height with a measuring tape. The felled trees were then cut into five-foot bolts (1.524 m) starting at ground level up to a top diameter of less than two inches (5.1 cm). Each bolt was measured for diameter inside and outside bark at the small end. In addition, the first bolt from the stump (i.e. butt bolt) was measured for midpoint diameter inside and outside bark. Cubic foot volumes for each tree to various upper diameters were obtained using Bailey's (1995) overlapping-bolts method. For the loblolly pine trees, age averaged 14 years and ranged from 9 to 26; dbh averaged 16.5 cm and range from 3.3 to 34.5; and total tree height averaged 13.1 m and ranged from 3.9 to 25.8. For the slash pine sample trees, age averaged 15 years and ranged from 9 to 27; dbh averaged 16.0 cm and range from 6.6 to 34.3; and total tree height averaged 13.7 m and ranged from 5.8 to 23.2 .

Total volume equation selection

A total volume equation [$V(D, H)$] is a necessary component both in the taper model (equation 16) and merchantable volume model (equation 21) in order to invoke the compatibility constraint (equation 7). In order to determine the most appropriate model for the total volume equation, 7 individual total volume equations given by Clutter et al. (1992) were considered. The logarithmic form,

$$V(D,H)=a_0 D^{a_1} H^{a_2},$$

was determined to be best for loblolly and slash pine based on goodness of fit statistics computed with our data.

Parameter estimation

The equation system derived above consists of three inter-dependent components, namely the total volume equation, the merchantable volume equation, and a taper equation. The total volume equation naturally arises as a special case included in the merchantable volume model.

With our data we considered the following two methods of fitting the system:

1. Estimate a_0, a_1, and a_2 independently of the other parameters using the total volume observations, and then substitute the estimated parameters into the system.
2. Estimate all 8 parameters simultaneously.

When all parameters including a_0, a_1, and a_2 are estimated simultaneously, one should use caution. Although it is theoretically correct that when h is equal to H (i.e. z=1) in our system the merchtanble volume model will convert to the total tree volume equation [i.e. $V_m = V(D,H)=a_0 D^{a_1} H^{a_2}$ when $h = H$], the observations of total volume can not be used directly in the parameter estimation process. The nonlinear parameter estimation algorithm makes use of logarithms of 1-z. This quantity will be zero when $h = H$ and the logarithm will be undefined. Consequently, these observations will be ignored in the parameter estimation process. However, it is not a good idea to use the estimated a_0, a_1, and a_2 to predict total volume without having the total volume observations involved in the parameter estimation proceses. We solve this problem by defining a dummy variable δ as:

$$d=\begin{cases} 1 & If \quad h=H \\ 0 & Otherwise \end{cases}$$

and rewrite the merchantable volume equation as

$$\mathrm{V_m} = (1-d)V_m(D,H,h,h_0) + d\,\mathrm{a}_0 D^{a_1} H^{a_2} \qquad , \qquad (23)$$

where the merchantable volume model, $V_\mathrm{m}(D,H,h,h_0)$, is defined by (21).
When h=H, δ = 1 and the first term in (23) reduces to zero. We reassign an arbitrary value smaller than 1 to z to avoid the logarithm of 0 and the second part of (23) is $\mathrm{a}_0 D^{a_1} H^{a_2}$, which is identical to the merchantable volume model in this case. When h is other than H, δ = 0 and the second term in (23) becomes zero. The first term in (23) is the merchantable volume model itself. Similarly, we may rewrite the taper model as

$$d=(1-d)d(D,H,h) \quad ,$$

where $d(D,H,h)$ is defined by equation (16). Now total volume observations can be used in the simultaneous parameter estimation process. Thus the total volume equation is a special case of the merchantable volume model with $h = H$, both in theory and in parameter estimation. We may predict total volume by $\hat{\mathrm{a}}_0 D^{\hat{a}_1} H^{\hat{a}_2}$, where $\hat{\mathrm{a}}_0, \hat{a}_1, \hat{a}_2$ are simultaneously estimated with the taper and the merchantable volume models. When using this methodology, the original data set should be checked carefully to make sure the observed values are logically correct, namely, when observed $h = H$, observed d

should be 0 and observed merchantable volume should be equal to observed total tree volume.

The compatible system developed above is very easy to encode in common software. We used the SAS MODEL procedure (SAS institute Inc. 1993) in which several parameter estimation methods are available. If error terms of the taper model and the merchantable volume model are independent and homogeneous, ordinary least squares estimation (OLS) may be applied, otherwise generalized least squares (GLS) methods are more efficient. Since an endogenous variable of one equation is not used as a regressor by another equation in our system, seemingly unrelated regression (SUR) may be applied. Two other approaches appropriate for our model system that are also available in the SAS MODEL procedure are the full information maximum likelihood estimation method (FIML) and generalized method of moments (GMM). GMM requires instruments. There is no standard way to set up instrumental variables and poor instruments may destroy the whole parameter estimating process. When the system of equations has heteroscedastic errors, GMM can be used to obtain efficient parameter estimates. FIML does not require instrumental variables, but it assumes that equation errors have a multivariate normal distribution. Since with our data the errors are almost normally distributed and the correlation between the error terms of the taper model and merchantable volume model are not so high [For instance, for loblolly pine $Cov(\varepsilon_d, \varepsilon_{Vm}) = 0.052$], there is not much difference in the results from OLS, SUR, and FIML. However, it was very difficult to get convergence with GMM. Considering the log likelihood values reported by FIML which can be used to evaluate the goodness of fit of the models, we considered the full information version of maximum likelihood estimation in the MODEL procedure preferable.

Results and discussion

Parameters in the system were estimated by two different approaches. In the first approach the logarithmic total volume equation was fitted independently using the total volume observations. Following this, those estimated parameters were substituted into the taper and merchantable volume models and the five remaining parameters in the system were estimated simultaneously. In the second approach all eight parameters in the system were estimated simultaneously using the technique described above for avoiding undefined logarithms. In the second approach parameters a_0, a_1 and a_3 in the total volume equation are estimated in such a way that they not only minimize the error squares in total volume, but also minimize the error squares for the taper and the merchantable volume models. The results from these two approaches are very close for both species (Table 1). This suggests that we may use a fitted total volume equation as a component of the system, making the compatible taper and merchantable volume (including total volume) system more flexible. For example, if total volume observations were unavailable in our data, we may use values from a volume table or some previously published equation as this component of the system. Moreover, if there are some convergence problems in simultaneously estimating all the parameters in the system for a given data set, fitting the total volume equation first may make things easier. However, one thing should be kept in mind. While independent estimation (for total volume) can lead to more accurate and precise prediction of total stem volume, it will undoubtedly increase the bias and standard error for the taper prediction, although the amount can be small. This is so because the system parameters in the total volume equation are obtained by minimizing the error sum of squares of total volume only not the errors in the taper model. These effects were very small for our data. For example, the root mean square error in upper-stem diameter for

Table 1 *Model coefficients for loblolly and slash pine from full information maximum likelihood estimation with two approaches. In approach 1, parameters a_0, a_1 and a_2 are estimated independently with total volume observations and the estimated parameters are substituted into the taper and merchantable volume model system, the five remaining parameters then are estimated simultaneously for the taper and merchantable volume models. In approach 2, the eight parameters are all estimated simultaneously by FIML.*

Loblolly Pine

	Alternative 1		Alternative 2	
parameter	estimates	Approx. std. error	Estimates	Approx. std. Error
a_0	.000019466	7.915E-7	.000019913	1.512E-7
a_1	1.851623	.01980461	1.946750	.0031296
a_2	1.29951	.02186523	1.185269	.0034586
b_1	.000011709	1.069E-5	.000010481	2.038E-7
b_2	.000038341	9.1818E-6	.000038139	1.256E-7
b_3	.000027844	4.675E-6	.000027594	1.087E-7
p_1	.076907	.0013795	.069669	.0016984
p_2	.566583	.0033846	.568960	.0033598
	Root MSE	R-Square	Root MSE	R-Square
Taper model:	0.9107 cm.	0.9806	0.89423 cm.	0.9813
Merch. Volume:	0.01544 m^3	0.9854	0.01519 m^3	0.9859
Log likelihood:	17496		17785	

Slash Pine

	Alternative 1		Alternative 2	
parameter	estimates	Approx. std. error	estimates	Approx. std. error
a_0	.000024127	1.40E-6	.000023132	2.265E-7
a_1	1.9903466	.028168	2.054709	.0043363
a_2	1.0740832	.0037033	1.017885	.0049481
b_1	.000010802	2.531E-7	.000010548	2.945E-7
b_2	.000040675	1.708E-7	.000040668	1.179E-7
b_3	.000030242	1.031E-7	.000030187	8.1831E-6
p_1	.071433	.0020414	.070196	.0023286
p_2	.541493	.0043988	.541673	.0043263
	Root MSE	R-Square	Root MSE	R-Square
Taper model:	0.80997cm.	0.9834	0.80486 cm.	0.9836
Merch. Volume:	0.01352 m^3	0.9871	0.01355 m^3	0.9870
Log likelihood:	14868		14918	

loblolly pine only increased from 0.8942 to 0.9107 cm (1.84%, Table 1) between the simultaneous estimation approach and the "independent" total volume approach. The corresponding increase was 0.63% for slash pine (Table 1).

For merchantable volume, the effects of independent volume estimation depend on how different the topmost bolt's taper is as compared to all other bolts in the tree. If the relationship between bolt dimensions and volume are similar for all bolts including the topmost, the accuracy and precision may increase with independent total volume estimation. For the slash pine data, the root mean squared error actually decreased by 0.3% from 0.01355 to 0.01352 m^3 (Table 1) when total volume was fitted independently. On the other hand, if the form is different for the topmost bolt even though error sum of squares is minimized for total volume the error sum of squares for the general merchantable volume may still increase. For the loblolly pine data, root mean square error in volume increased by 1.69% from 0.01519 to 0.01544 m^3 with independent estimation of total volume (Table 1). Since both the taper and the merchantable volume equations share parameters in the total volume equation, if total volume observations are available we recommend the simultaneous estimation approach. As further support of our recommendation, we note that the log-likelihood values of the system increased by 289 (1.65%) for loblolly pine and 50 (0.34%) for slash pine with simultaneous estimation.

All figures, tables, and discussion that follow will be based on the parameters estimated in the simultaneous approach. When all parameters in the system are estimated simultaneously, it is necessary to introduce a dummy variable so that total volume observations are used in the parameter estimation process. For example, without such a technique, for loblolly pine, the average bias of the total stem volume increased from 0.00085 to 0.00102 m^3 and standard errors of estimate (SEE) increased from 0.0178 to 0.0183 m^3 (2.35% increase).

Since all parameters in the system have a specific meaning, it is interesting to examine their estimated values. First, the estimated p_1 and p_2 show that for both loblolly and slash pine one of the inside-bark inflection points occurs near dbh and another is just above one half of total tree height. Specifically, the two inside-bark inflection heights for loblolly pine are around 7% and 57% of total stem height. Thus, of the three sections segmented by the two inflection points the first is very short as a percentage of total stem length.

The "normalized" form factors for the three segments (lowest to highest) as found by dividing the estimated form factors b_i by k are 0.134 , 0.486, and 0.351 for loblolly pine and 0.135, 0.518, and 0.384 for slash pine. Both show that the inside-bark form factors are smallest at bottom, largest in the middle and moderate at the top. This makes intuitive sense if the relative positions of inflection points are taken into account. These form factors also compare quite remarkably to 0.250, 0.500, and 0.333 for a neiloid, paraboloid, and cone, respectively.

Predicted taper curves both for loblolly and slash pine trees with DBH=30.48 cm (12 in.) and total tree height=22.86 m (75 ft.) are quite reasonable in appearance (Figure 1). Goodness of fit statistics show that the compatible model system fit both the loblolly pine and slash pine data reasonably well. For example, more than 98% of the variation about the mean values of d and V_m is explained by the model system for both species.

To test the effectiveness of the system more closely, biases and the standard errors of estimates for inside bark upper stem diameter and merchantable volume estimation (including total volume, also inside bark) were determined by dbh class (Table 2), total main stem height class (Table 3), and at several heights above ground level

(Table 4). The bias and standard errors of estimates (SEE) in the tables are calculated with

$$Bias = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)\Big/n ; \qquad SEE = \sqrt{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 \Big/(n-p)}$$

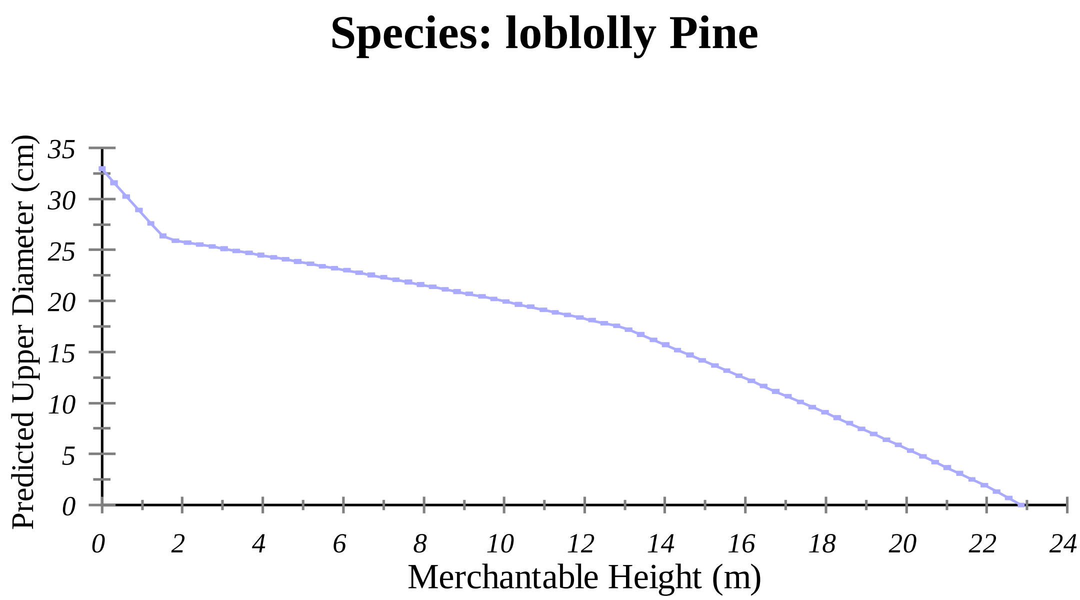
where Y_i is the measured value, $\widehat{Y}_i$ is the estimated value, n is the number of observations, and p is the number of parameters estimated. Since the eight parameters in our model system are shared by the taper and merchantable volume equations, *p* is 4 in this case. Percent bias and percent SEE in Tables 2, 3, and 4 are based on the average observed value of the class. The system performs well for both loblolly and slash pine. The bias and SEE are small and generally stable across DBH and total height classes. There is no noticeable trend (in percentage for SEE) from ground to the top, which indicates that the segmented-stem model reasonably reflects the butt part as well as the top part of the tree stems. When height from ground is 100%, namely merchantable height equals total stem height, the estimated diameters are always 0 and merchantable volume is identical to the total volume in our compatible system. Bias and SEE of total volume (relative height 100%, Table 4) for loblolly pine are 0.00085 m^3 (0.64%) and 0.0178 m^3 (13.37%). For slash pine they are 0.0005 m^3 (0.38%) and 0.0168 m^3 (12.46%) (Table 4).

A visual comparison between the performance of describing stem profile from ground to top with the segmented-stem taper system as opposed to a constant form factor model (i.e. assume $b_1 = b_2 = b_3$) shows some advantages in the segmented approach (Figure 2). The segmented-stem model is better than the constant form model in the following sense:

1. Bias and SEE of taper and merchantable volume estimation are smaller for the segmented model. A formal likelihood ratio test concludes a significant improvement from constant form factor to segmented form factor model system[1].
2. Bias and SEE of taper estimation are more stable from the bottom to the top of the bole for the segmented model. The constant form factor model tends to have larger bias and SEE for the taper around inflection points since the constant form may not sufficiently describe the stem profile near the butt.
3. The bias and SEE of the total stem volume are also smaller for the segmented model when parameters in total volume equation are estimated simultaneously in the system, as we did. If the parameters in the total volume equation are pre-estimated with total volume observations, the total volume estimations with the segmented and constant form factor models will be the same due to the compatibility constraint used in the system. But, increase in the bias and SEE of upper diameter estimation due to the independent estimation of parameters in the volume model will be larger for the constant form model.

[1] The likelihood ratio statistics for loblolly pine and slash pine are respectively 3306 and 2251, both with 3 degree of freedom.

Figure 1. Example stem profiles with the segmented-stem taper models for loblolly and slash pine trees with dbh of 30.48 cm and total height of 22.86 m.



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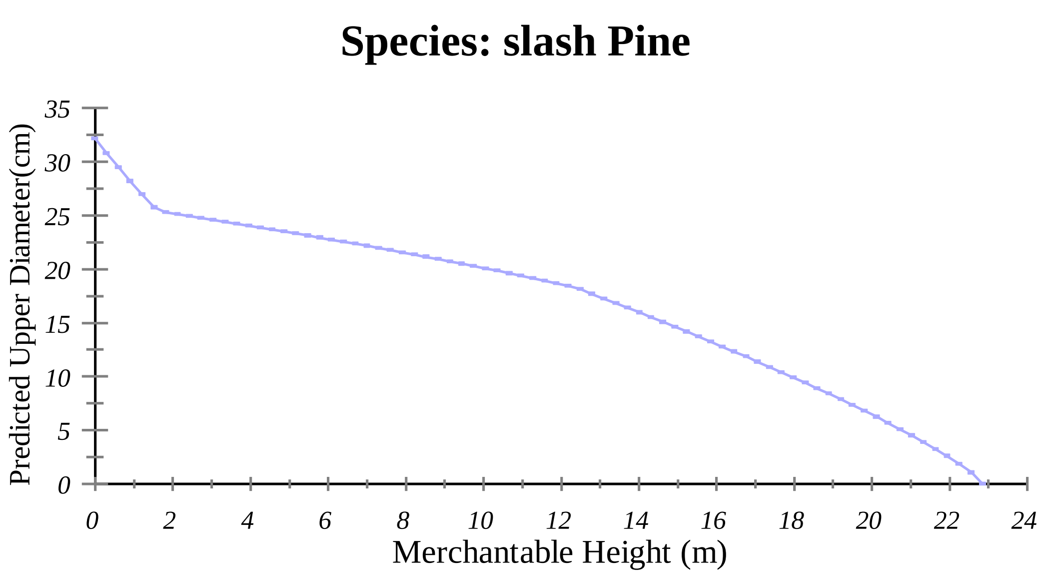


Table 2 ***Biases and standard errors of estimates by DBH classes for inside bark diameter and merchantable volume for the loblolly (L) and slash (S) pine data***

			Upper diameter				Merchantable Volume			
	DBH class		Bias		SEE		Bias		SEE	
Species	cm.	n	cm.	%bias	cm.	%SEE	m^3	%bias	m^3	%SEE
L	<9.0	605	0.121	2.75	0.444	10.09	0.00072	8.46	0.0014	16.79
L	9.0-10.9	807	0.062	1.07	0.526	9.02	0.00074	3.95	0.0025	13.13
L	11.0-12.9	1211	-0.012	-0.17	0.603	8.89	0.00038	1.29	0.0036	12.35
L	13.0-14.9	1322	-0.057	-0.73	0.716	9.08	0.00038	0.89	0.0053	12.40
L	15.0-16.9	1361	0.009	0.10	0.772	8.55	0.00165	2.72	0.0069	11.27
L	17.0-18.9	1535	-0.019	-0.19	0.903	9.02	0.00159	1.91	0.0094	11.24
L	19.0-20.9	1251	-0.115	-1.03	0.950	8.51	0.00020	0.18	0.0133	11.76
L	21.0-22.9	1059	-0.026	-0.21	0.985	7.89	0.00176	1.14	0.0152	9.91
L	23.0-24.9	885	-0.020	-0.15	1.050	7.77	0.00072	0.38	0.0185	9.63
L	25.0-26.9	586	-0.103	-0.71	1.061	7.27	-0.00230	-0.97	0.0169	7.12
L	27.0-28.9	408	0.093	0.58	1.311	8.10	0.00199	0.62	0.0272	8.53
L	29.0-30.9	242	-0.231	-1.36	1.512	8.94	-0.01019	-2.78	0.0454	12.40
L	31.0-32.9	179	0.254	1.38	1.456	7.88	0.00543	1.20	0.0463	10.24
L	33.0-34.9	89	-0.058	-0.30	1.759	9.18	0.00408	0.82	0.0564	11.35
	Total:	11540	-0.018	-0.06	0.859	8.62	0.00067	1.62	0.011	11.43
S	<9.0	710	-0.039	-0.88	0.487	10.96	0.00018	1.75	0.0015	14.73
S	9.0-10.9	713	-0.108	-1.91	0.530	9.38	-0.00017	-0.90	0.0023	11.79
S	11.0-12.9	914	0.015	0.22	0.624	9.00	0.00037	1.15	0.0038	11.72
S	13.0-14.9	918	0.045	0.57	0.714	8.98	0.00092	1.93	0.0060	12.49
S	15.0-16.9	804	-0.084	-0.93	0.750	8.38	-0.00058	-0.93	0.0076	12.26
S	17.0-18.9	1073	0.042	0.42	0.775	7.61	0.00140	1.55	0.0095	10.48
S	19.0-20.9	869	-0.036	-0.32	0.896	7.95	-0.00004	-0.04	0.0127	10.57
S	21.0-22.9	707	0.024	0.19	0.930	7.44	0.00206	1.26	0.0152	9.38
S	23.0-24.9	344	-0.156	-1.15	0.910	6.69	-0.00239	-1.17	0.0173	8.47
S	25.0-26.9	768	-0.031	-0.21	1.048	7.05	-0.00041	-0.16	0.0235	9.09
S	27.0-28.9	389	0.143	0.88	1.173	7.27	0.00392	1.25	0.0332	10.56
S	29.0-30.9	114	-0.078	-0.46	0.907	5.28	-0.00362	-1.00	0.0212	5.85
S	33.0-34.9	28	-0.886	-4.79	1.380	7.46	-0.02699	-6.29	0.0322	7.51
	Total:	8351	-0.017	-0.25	0.783	8.31	0.00034	0.58	0.0108	11.12

Note: The total for bias, %bias, SEE and %SEE are calculated as overall mean.

Table 3 ***Biases and standard errors of estimates by height classes for inside bark diameter and merchantable volume for the loblolly (L) and slash (S) pine data***

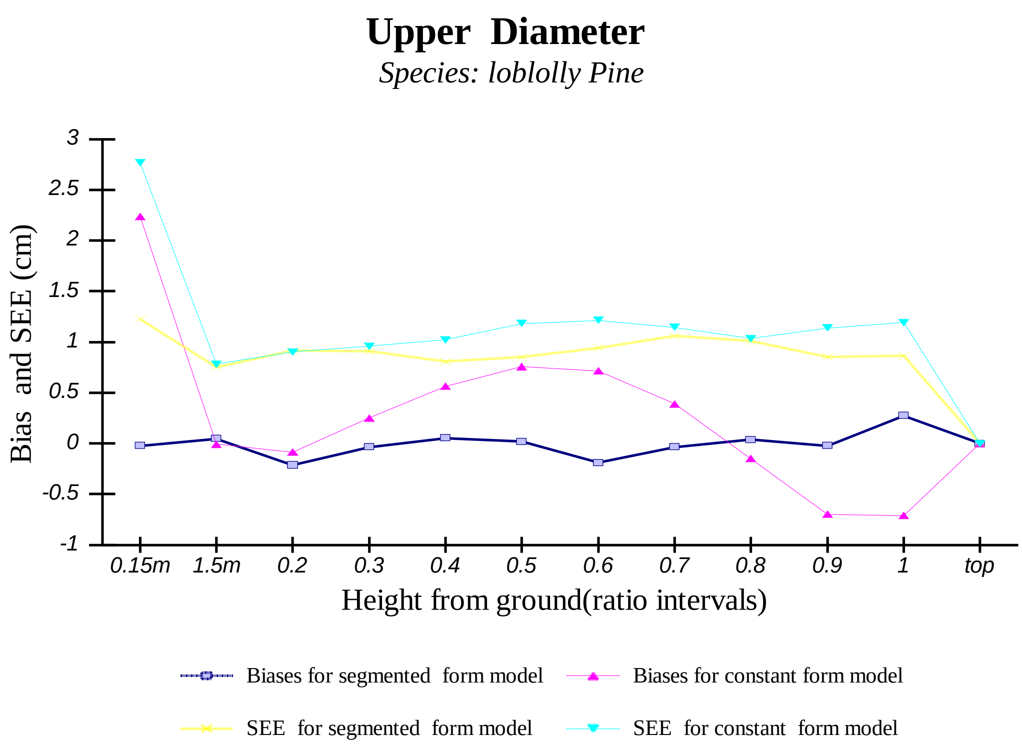
	total tree		Upper diameter				Merchantable Volume			
	height class		Bias		SEE		Bias		SEE	
Species	m	n	cm	%bias	cm	%SEE	m^3	%bias	m^3	%SEE
L	<6.0	53	0.157	4.17	0.520	13.78	0.00058	13.54	0.0011	25.31
L	6.0-7.9	516	0.135	2.69	0.592	11.77	0.00122	11.04	0.0030	26.88
L	8.0-9.9	1270	-0.015	-0.22	0.682	10.13	0.00128	5.04	0.0034	13.30
L	10.0-11.9	1857	-0.054	-0.68	0.712	8.90	0.00136	3.21	0.0048	11.43
L	12.0-13.9	2250	-0.062	-0.68	0.833	9.14	0.00105	1.59	0.0072	10.89
L	14.0-15.9	1926	-0.082	-0.77	0.883	8.32	0.00002	0.02	0.0103	9.99
L	16.0-17.9	1588	-0.025	-0.21	0.966	8.06	0.00055	0.37	0.0172	11.54
L	18.0-19.9	1083	0.037	0.28	1.047	7.76	0.00011	0.05	0.0179	8.53
L	20.0-21.9	664	-0.003	-0.02	1.200	7.67	-0.00239	-0.77	0.0319	10.32
L	22.0-23.9	234	0.205	1.18	1.572	9.05	-0.00346	-0.83	0.0525	12.65
L	24.0-25.9	99	0.832	4.47	1.525	8.18	0.02011	4.08	0.0373	7.56
	Total:	11540	-0.018	-0.20	0.873	8.84	0.00067	1.97	0.0117	11.71
S	<8.0	246	-0.037	-0.80	0.507	11.11	0.00030	3.14	0.0014	14.04
S	8.0-9.9	683	0.030	0.53	0.572	10.23	0.00083	4.90	0.0027	15.86
S	10.0-11.9	1363	-0.035	-0.50	0.641	9.24	0.00061	1.91	0.0042	12.91
S	12.0-13.9	1384	-0.022	-0.28	0.684	8.57	0.00046	0.94	0.0069	13.86
S	14.0-15.9	1591	-0.022	-0.21	0.824	8.18	0.00058	0.64	0.0097	10.60
S	16.0-17.9	1343	-0.028	-0.24	0.934	8.11	0.00022	0.16	0.0158	11.67
S	18.0-19.9	884	-0.038	-0.28	0.928	6.71	-0.00071	-0.32	0.0201	9.08
S	20.0-21.9	765	-0.005	-0.03	0.998	6.52	-0.00166	-0.58	0.0236	8.20
S	22.0-23.9	92	0.391	2.32	1.238	7.34	0.01534	4.02	0.0432	11.34
	Total:	8351	-0.017	-0.20	0.790	8.34	0.00034	1.06	0.0112	11.85

Table 4 *Biases and standard errors of estimates at several heights from ground level for inside bark diameter and merchantable volume for the loblolly (L) and slash (S) pine data*

			Upper diameter				Merchantable Volume			
Species	Height from ground	n	Bias cm	%bias	SEE cm	%SEE	Bias m^3	%bias	SEE m^3	%SEE
L	0.1524 m	1280	-0.025	-0.15	1.225	7.24	0.00000	0.00	0.0000	0.00
L	1.524 m	1280	0.047	0.35	0.754	5.55	0.00153	5.35	0.0049	16.96
L	10% - 20%	366	-0.214	-1.16	0.912	4.96	-0.00099	-0.98	0.0097	9.61
L	20% - 30%	999	-0.036	-0.25	0.911	6.39	0.00010	0.12	0.0093	10.81
L	30% -40%	1084	0.053	0.42	0.809	6.38	0.00019	0.18	0.0112	10.77
L	40% - 50%	1087	0.020	0.17	0.850	7.36	0.00048	0.38	0.0141	11.30
L	50% -60%	1111	-0.190	-1.87	0.942	9.28	0.00086	0.60	0.0165	11.51
L	60% - 70%	1090	-0.036	-0.44	1.059	12.74	0.00010	0.06	0.0187	12.18
L	70% - 80%	1057	0.039	0.62	1.010	15.99	0.00091	0.53	0.0215	12.63
L	80% - 90%	803	-0.025	-0.58	0.854	20.03	0.00171	0.85	0.0240	11.89
L	90%-100%	103	0.272	8.82	0.862	27.94	0.00666	2.12	0.0334	10.65
L	100%	1280	0.000	0.00	0.000	0.00	0.00085	0.64	0.0178	13.37
	Total:	11540	-0.018	-0.11	0.834	8.63	0.00066	0.89	0.0133	11.02
S	0.1524 m	871	-0.013	-0.08	1.164	7.17	0.00000	0.00	0.0000	0.00
S	1.524 m	871	-0.135	-1.04	0.696	5.38	0.00062	2.43	0.0030	11.71
S	10% - 20%	287	0.011	0.06	0.903	5.05	-0.00129	-1.48	0.0069	7.99
S	20% - 30%	735	0.061	0.44	0.807	5.88	-0.00041	-0.52	0.0077	9.76
S	30% -40%	797	0.061	0.49	0.823	6.60	0.00057	0.56	0.0100	9.82
S	40% - 50%	750	-0.062	-0.55	0.823	7.30	0.00052	0.44	0.0124	10.49
S	50% -60%	794	-0.224	-2.23	0.863	8.58	-0.00009	-0.07	0.0147	10.76
S	60% - 70%	805	0.069	0.81	0.840	9.94	0.00057	0.39	0.0168	11.29
S	70% - 80%	752	0.083	1.24	0.783	11.72	0.00018	0.11	0.0187	11.30
S	80% - 90%	678	-0.007	-0.15	0.731	15.89	0.00061	0.34	0.0201	11.13
S	90%-100%	140	0.013	0.40	0.663	19.84	0.00404	1.52	0.0257	9.67
S	100%	871	0.000	0.00	0.000	0.00	0.00050	0.38	0.0168	12.46
	Total:	8351	-0.017	-0.11	0.753	7.74	0.00032	0.38	0.0118	9.73

Note: The height ratio intervals are right-side excluded. For example, 40% - 50% means that h/H is in the interval from 0.4 to 0.5, not including 0.5. The biases and SEEs for total volume can be referenced in the box.

Figure 2. Comparisons of the biases and standard errors of estimate (SEE) for inside-bark upper stem diameter and merchantable volume at different heights for the segmented model and the constant form model for loblolly and slash pine.



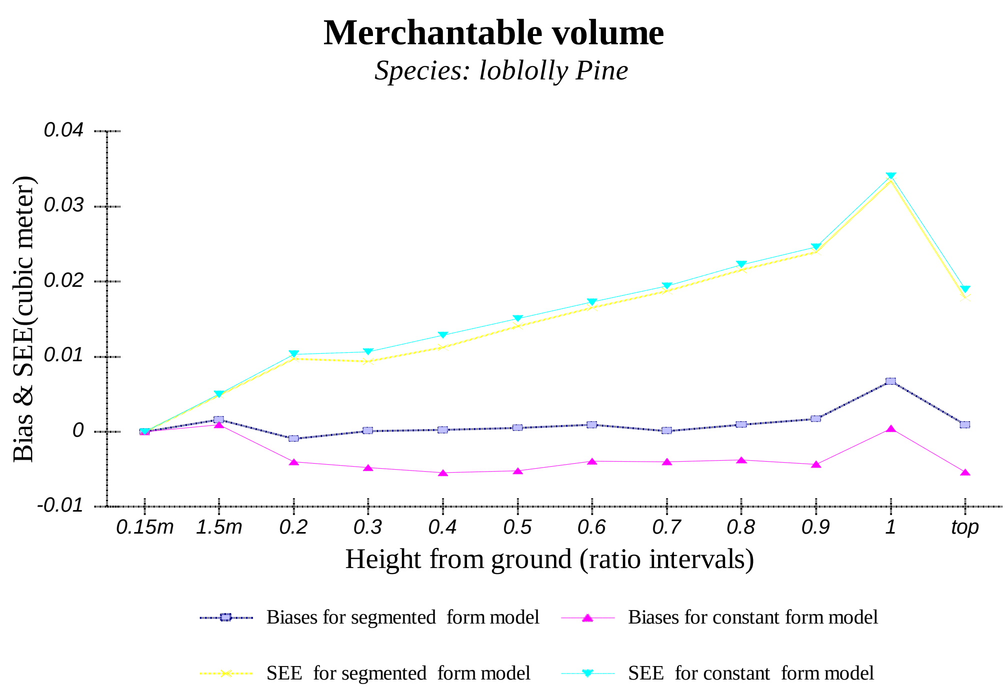
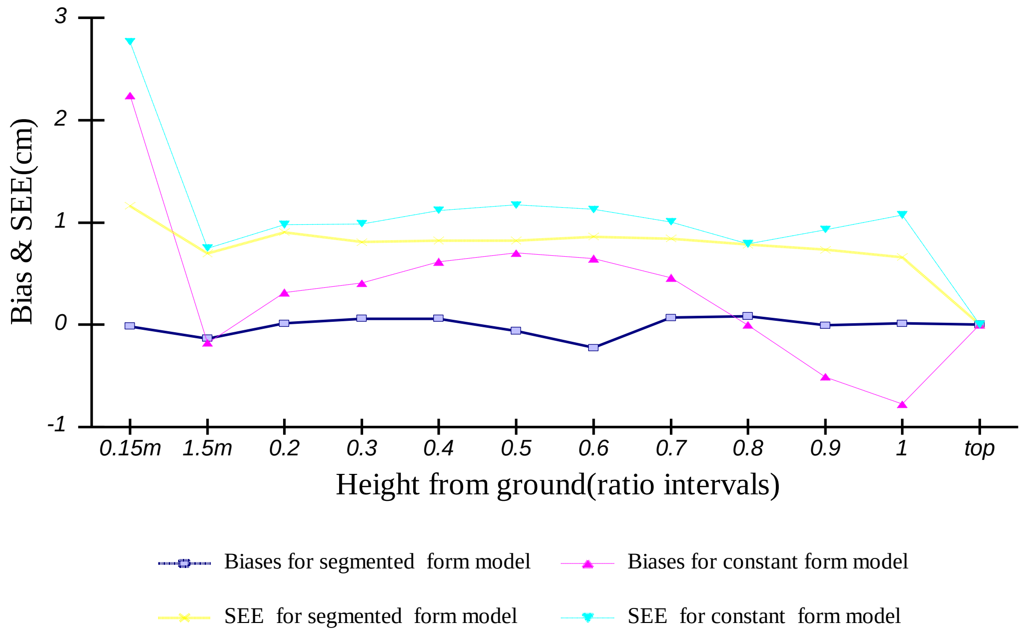


Figure 2 cont. Comparisons of the biases and standard errors of estimate (SEE) for inside-bark upper stem diameter and merchantable volume at different heights for the segmented model and the constant form model for loblolly and slash pine.

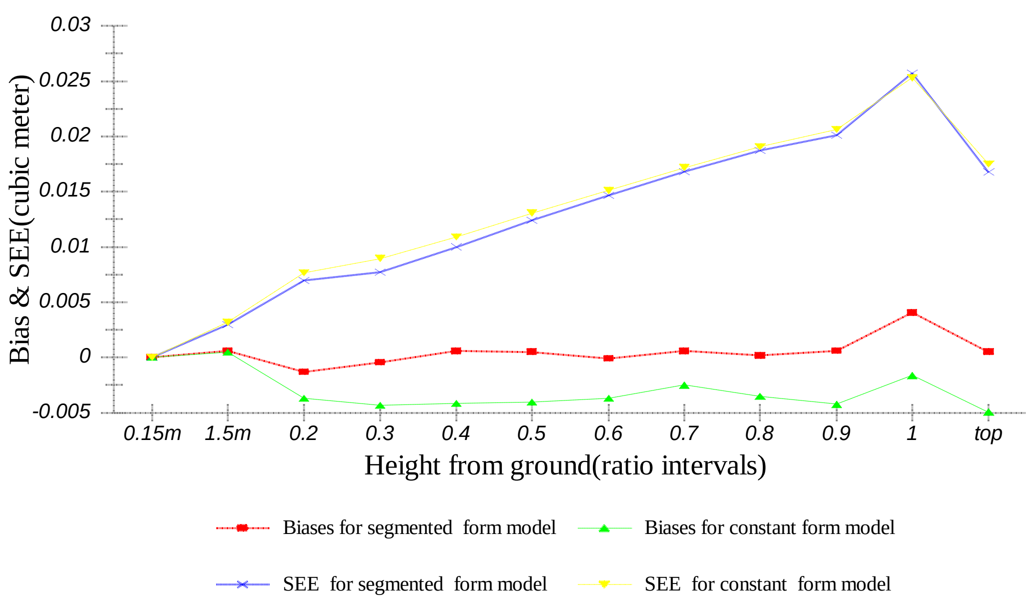
Upper Diameter

Species: slash Pine



Merchantable Volume

Species: slash Pine



The performance for merchantable volume and total volume prediction with the constant form model is very close to the segmented model. This is because the total volume compatibility constraint is also included in the constant form model. The constant form factor model is intrinsically linear, i.e., the models can be linearized by logarithm transformation of the dependent variables (d and V_m). So linear regression techniques can be used to estimate the parameters. Actually, as shown before, the constant form factor model is a special case of the segmented model (when $b_1 = b_2 = b_3$), and thus the segmented model is readily adjusted in real applications. For example, if we were only interested in the middle part of the bole and tried to model a small sample of data in a local application, the constant form model may be adequate.

The segmented model (equation 16) belongs to a large family of segmented polynomial models (Anderson and Nelson 1975, Gallant and Fuller 1973). By logarithmic transformation, the taper model (equation 16) can be viewed as a polynomial with respect to the term log(1-h/H). All other transformed independent variables, for example log(D), can be expressed as some polynomial form. Similarly, the tip volume model (total volume - merchantable volume) also can be regarded as a segmented polynomial model with respect to the term log(1-h/H). If the "joint points" are known the segmented polynomial models are linear, otherwise they are nonlinear. Except for the two inflection points, the "joint points" include another "point" which serves as our total volume constraint. Therefore, except for the case of constant form ($b_1 = b_2 = b_3$), our segmented models are also intrinsically linear for the case when the relative inflection points p_1 and p_2 are known and three form factors are proportional to each other with these proportions also known.

For the segmented model, the constraints are usually imposed in such a way that the segments: ① meet each other at inflection points and ② meet smoothly. The first constraint requires that adjacent functions at an inflection point be equal so the curve is continuous at the inflection points. The second constraint requires the first derivative of adjacent functions be equal at the inflection point (smoothness constraint). In this paper, since there is not much practical significance for smoothness at inflection points and the smoothness constraint will unavoidably restrict the behavior of the parameters in parameter space, we only use the first constraint (continuity) in our model derivation. For example, if the second constraint is also imposed, except for equations (5) and (6), another two equations would need to be satisfied,

$b_2(k - b_1) = b_1(k - b_2)$ and

$b_3(k - b_2) = b_2(k - b_1)$,

which unnecessarily restricts the parameter space.

The smoothness constraint is meaningless without the continuity constraint. Based on the proposition that differentiability of a function is sufficient but not necessary to continuity, it is common in forest literature to only impose the smoothness constraint on segmented models. This takes for granted that continuity is also guaranteed. This is incorrect because the proposition concerning continuity and differentiability is true only for one distinct function. Segmented models can be differentiable at the inflection points without meeting each other let alone being continuous. Smoothness only assures that the slopes of the curves are parallel to each other and does not assure that they will meet. To see this more clearly, suppose we only impose the smoothness constraint on inflection point p_1 in our taper model. After a little algebra, instead of equation (5), the following equation obtains:

$$c_2 = c_1[H(1-p_1)]^{(b_2-b_1)k/2b_1b_2} \; b_2(k-b_1)/[b_1(k-b_2)]$$

When compared to equation (5), after only imposing the smoothness constraint, the two sections can not meet at p_1 (let alone to be continuous) unless $b_2(k - b_1) = b_1(k - b_2)$.

The model system developed in this paper is based on the assumption that there are two inflection points on a bole. But, as mentioned before, it is not necessary to put such a condition on our model derivations. For example, if there are three inflection points on a bole, our model derivation can be reasonably adjusted to include three inflection point parameters (p_1, p_2 and p_3) in the model. There would be four coefficients of integration (c_1 to c_4) and we would use three of them to satisfy the continuity constraints on three inflection points and one to satisfy the volume compatibility constraint. The whole model derivation would be very similar to what we show above.

From equation (16) we can solve for merchantable height, *h*, explicitly. Thus, the same parameter estimates may be used to predict *h* for a given *d*. Even though this approach of using the fitted taper curve to predict merchantable height from upper-stem diameter is common in forestry, there are some statistical problems with it. This is so because the taper model was developed based on the assumption that observed upper diameter contains a random component and merchantable height is known without error. When the assumption is reversed, the error distribution of height is not identical to that of diameter and this reversal will unavoidably increase the prediction error. Also, the parameters in the fitted taper model are estimated by minimizing the sum of squared errors of diameters not heights. The segmented model was derived based on the assumption that height is non-random and diameter random, so the models are constrained to make the predicted diameters meet each other at the inflection points to insure continuity. This does not imply that predicted heights are continuous at the inflection points when heights are considered as random and diameters as fixed. Furthermore, for the compatible volume constraint, we should compute the total volume from the merchantable height curve by integrating diameter, not height, if diameter is indeed fixed and height random. A reasonable alternative to this problem is to develop a compatible merchantable height model using similar logic as for the taper model and then fit the system simultaneously.

Our compatible taper and merchantable volume models are readily transformed into compatible diameter and ratio volume models. If we define $R_d = d/D$ as the diameter ratio and $R_v = V_m/V(D, H)$ as the volume ratio, then the compatible ratio model can be written as:

$$R_d = \sqrt{\frac{a_0}{c^*} D^{a_1-2} H^{a_2-1} a_1^{I_1+I_2} a_2^{I_2} (1-z)^{(k-b)/b}}$$

$$R_v = [b_1 t_0 + (I_1 + I_2)(b_2 - b_1)t_1 + I_2(b_3 - b_2)a_1 t_2 - a_1^{I_1+I_2} a_2^{I_2} b(1-z)^{k/b}]/c^*,$$

where $c^* = b_1(t_0 - t_1) + b_2(t_1 - a_1 t_2) + b_3 a_1 t_2$.

It is easy to show, at the top, h=H (i.e., z=1), R_d=0, and R_V =1; and at the stump, h=h_0 (i.e., z=p_0), R_V =0;

It should mentioned that the correlation among the bolts of an individual tree was not considered in above approach. This obviously does not mean that this correlation is not important. It is potential that the prediction of stem profile will even more improved if bolts correlation is reasonably traced(see Williams and Reich 1996).

Summary

Based on an exponential tip volume model with variable form factors, a segmented-stem differential equation for taper was introduced and a special case solved based on the assumption that a bole is segmented into three sections by two inflection points with a constant and distinct form factor in each section. The resulting segmented-stem taper model and merchantable volume model are compatible with each other and both are also compatible with total stem volume. All parameters are shared by the components in this compatible system. Simultaneous estimating techniques are introduced so that all information in the data is used for parameter estimation. The segmented-stem taper model and merchantable volume model, including the total volume equation, were evaluated with 1280 loblolly pine trees and 871 slash pine trees. The system performed well for the two data sets. There is an obvious improvement with respect to predicting a stem profile with the segmented-stem models compared to a constant form factor model. Even though the model in this paper is derived based on two inflection points on a bole, the constant form factor model (0 inflection points) and one inflection point segmented model can be obtained as special cases by substituting appropriate parameters directly into the current model. In summary the segmented-stem model has the following properties:

1. The system is compatible in the sense of:
   - ① The total volume obtained by integration of the taper model is equal to that computed by the total volume equation. Thus the taper model and total volume equation are compatible.
   - ② The estimated merchantable volume for a given merchantable height is equal to that obtained by integrating the taper model. Thus the taper model and merchantable volume model are compatible.
   - ③ When $h=H$, the merchantable volume model is algebraically identical to total volume model, therefore, the merchantable volume model is compatible with the total volume model and the total volume model is just a special case of the merchantable volume model.
   - ④ When $h=H$, $d=0$ (i.e., diameter at the top of the tree is zero).
   - ⑤ When, $h = h_0$, $V_m = 0$ (i.e., volume at the stump height is zero).
2. The segmented-stem taper model is continuous at inflection points.
3. Constant form factor model for taper and its corresponding merchantable volume equation are a special case of the two-inflection-point model. A segmented model with only one join point is another special case that can also be obtained from our segmented-stem system.
4. The segmented-stem taper model and tip volume model belong to the segmented polynomial family, which is intrinsically linear if the "joint points" are known, otherwise, they are nonlinear.
5. Every parameter in the system has specific meaning, so the estimated parameters themselves are informative.
6. A total volume equation can be used as input into the system. This makes the system flexible in application.
7. The system is easy to implement on a computer and simultaneous estimates make a reasonable compromise among the components in the system in the process of minimizing the sum of squared errors. Fitting methods that account for the correlation between the components (if any exists) may improve the effectiveness of the system.
8. The segmented-stem models describe the stem profile quite well.
9. It is easy to transform the current compatible model into a compatible volume ratio model.

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