Two Approaches to Improving Inventory Projection Equations on Examples of Published Models

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Abstract

This article focuses on mathematical aspects of equation development in growth and yield modeling. Presented are two examples of, 3-parameter, dynamic site equations derived from, 5- parameter, fixed base-age site index and height growth equations. These are truly base-age invariant initial-condition difference equations that compute appropriate heights at all base-ages and provide compatible height and site index values from one common equation. Although having fewer parameters, they can model broader selection of curves than the original equations. The new equations are recommended for all situations in which the original equations would be applicable. The displayed methods are recommended for consideration in developments of all recursive and otherwise implicitly defined equations, such as, the site index and height growth models.

Keywords: site index models, dynamic equations, initial condition difference equations, base-age invariant, model constraining.

Background

Forest inventory updates and projections frequently rely on self-referencing (Northway 1985) functions, such as, initial condition difference equations. In addition to their general parameters, these equations require snapshot observations of inventory-related information in order to compute their predictions. They are used to model various forest characteristics that depend on productivity sites. Examples of such characteristics are height, diameter, basal area, volume, biomass, carbon, and trees per unit area. Of all models based on implicitly defined equations, the most popular are "site index models" used for height predictions. These usually denote site-dependent models computing heights as functions of ages and site indexes. They are fitted to height-age data from different productivity sites and can be expressed by either dynamic or static equations.

The dynamic site equations are essentially initial condition difference equations that use variable base-ages. Selection of base-ages does not affect any predictions of dynamic equations. Bailey and Clutter (1974) have formalized this property as base-age invariance and have presented an approach to derivation of base age invariant site equations that became known as the algebraic difference approach. This approach consists of designating, in a considered function of age, one site-dependent parameter and substituting it with its initial-condition solution. The initial condition difference equations, in different forms, have been in use already even before Newton[1] invented differential calculus. An early example of such an equation is the Kepler's[2] second law, which is, $A_2 - A_1 = .5h(t_2 - t_1)$. This equation is equivalent to: $A_2 = A_1 + .5h(t_2 - t_1)$, and substituting in it, $\ln H$ for A, and b for $.5h$, results in: $\ln H_2 = \ln H_1 + b(t_2 - t_1)$, which is the anamorphic site equation used to formalize the concept of base-age invariance in Bailey and Clutter (1974). The dynamic site equations have the general forms of: $Y_2 = f(t_2, t_1, Y_1, \boldsymbol{\beta})$, or: $Y = f(t, t_0, Y_0, \boldsymbol{\beta})$, where Y_2 and Y are the function values at t_2 and t, Y_1 and Y_0 are the initial conditions defined as the functions values at t_1 and t_0, and $\boldsymbol{\beta}$ is the vector of model parameters. The dynamic site equations define both height-growth and site index models as special cases of the same equation. Yet, dynamic forms of site equations are relatively scarce. Out of some few hundred publications on site models only a couple dozens use dynamic equations and they are those of Amaro *et al.* (1998); Bailey and Clutter (1974); Bégin and Schütz (1994); Borders *et al.* (1984 and 1988); Cao *et al.* (1993 and 1997); Cieszewski and Bella (1989); Cieszewski *et al.* (1996); Clutter *et al.* (1983); Clutter *et al.* (1984); Elfving and Kiviste (1997), Eriksson *et al.* (1997) and Kiviste (1997 and 1998); Lappi

[1] Sir Isaac Newton. 1642-1727. English mathematician and scientist who invented differential calculus and formulated the theories of universal gravitation, terrestrial mechanics, and color.

[2] Johannes Kepler. 1571-1630. German astronomer and mathematician. Considered the founder of modern astronomy, he formulated three laws to clarify the theory that the planets revolve around the sun.

and Bailey (1988); Lenhart (1968 and 1972); McDill and Amateis (1992); and Ramirez *et al.* (1987).

The static site equations have been originated at the beginning of XX century. Presently, they are, by far, the most popular forms of site equations and they have the following general form: $H = f(t, S, \boldsymbol{\beta})$, where H is the height at age t, $\boldsymbol{\beta}$ is the vector of model parameters, and S is a fixed base-age site index, such as, for example, a height at the fixed base-age of 50 years.

Monserud (1984) defines two static site equations as: 1) "site index model" computing a site index from height and age; and 2) "height growth model" computing height from ages and site indexes. I adopt these definitions in referring to the static site models. In referring to any base-age invariant (Bailey and Clutter 1974) site equation, which directly estimates height and site index from any other height and age, I use the term dynamic site equation.

I consider here the two site equations discussed by Monserud (1984). The site index equation for estimating S from H and t is:

$$S = \alpha - \beta \ln^2 t + \gamma t \ln t + \delta H + \zeta \frac{H}{t} \tag{1}$$

where α, β, γ, δ and ζ are the model parameters estimated through regression analysis and the base age of S is 50 years.

The height growth equation estimating H from t and S is:

$$H = \frac{\alpha S^\beta}{1 + e^{\gamma - \delta \ln t - \zeta \ln S}} \tag{2}$$

where all symbols are as previously defined. All the parameters are estimated independently for each of the equations (1) and (2).

Both equations were fitted to the same stem analysis data. Equation (1) was fitted with site indexes, i.e., heights at age 50, as the dependent variable and all other heights and ages defined as the independent variable. Equation (2) was fitted with ages and site indexes, i.e., heights at age 50, as the independent variable and all other heights defined as the dependent variable.

To compare eq. (1) with eq. (2) the former was represented graphically by its inverse function. To do this, eq. (1) was solved for height H and applied with the height on the left hand site as a direct inverse function of eq. (1) with its original parameters. The inverse function of eq. (1) is

$$H = \frac{t(\beta \ln^2 t - \gamma t \ln t + S - \alpha)}{\delta t + \zeta} \tag{3}$$

where all symbols and parameter values are the same as in eq. (1).

The inland Douglas fir stem analysis data represented five habitat types. Due to growth pattern similarities, some habitats were pooled together into larger groups, which resulted in three sets of model parameters for three respective habitat groups. One of the parameter sets was also applicable for a general model of aggregate habitats.

Models defined by eq. (1) and (2) have been applied and cited in over 50 publications. These include different adaptations and variations of the height growth model (2). Examples of more popular (cited by others more than five times) articles citing this work include: Alemdag (1988), Biging (1985), Cieszewski and Bella (1989), McDill and Amateis (1992), Newnham (1988), Smith and Watts (1987), Vanclay (1992), Walters *et al.* (1989) and Wykoff (1990). Examples of eq. (2) adaptations include: Nussbaum (1996), Goudie (1984), AFS (1985), BCMF (1991), Thrower (1992), Thrower *et al.* (1994), Nigh (1997 and 1998) and Nigh and Courtin (1998).

Equation (1) can be used in its current or inverse form (eq. (3)) for developing either site index or height growth models. However, eq. (2) cannot be easily solved for site index and is usually applied with other means of obtaining compatible site indexes. At times, it is used together with eq. (1) as a companion site index model. Neither eq. (1) nor eq. (2) are conditioned for equality between heights at base age and site indexes.

The objectives of this article are to:

1) improve eqs. (1) and (2) with emphasis on succeeding them with dynamic site equations defining the same curve shapes; and

2) demonstrate advantages of the new improved equations in terms of parsimony, flexibility and other desirable characteristics, such as, the equality between site indexes and heights at base ages.

A Dynamic Site Equation for the Site Index Model

One of the oldest equations used in the context of growth and yield modeling is the Hosfeld's (1822) modified half-saturation function:

$$Y = \frac{t^\delta \theta}{\zeta_1 + t^\delta} \tag{4}$$

where: Y is the response variable, t is the independent variable, and the model parameters are θ, ζ_1, and δ. Assuming hereafter in this section, in eq. (4), that $\delta = 1$ and θ is the site parameter (Bailey and Clutter (1974), for an anamorphic model, results in the following site-dependent height equation:

$$H = \frac{t\,\theta_i}{\zeta_1 + t}$$

where θ_i is the site-specific parameter and ζ_1 is the global parameter. On the other hand, selecting ζ_1 as the site-specific parameter and θ as the global parameter would result in a polymorphic site equation with a single asymptote. One approach, allowing to develop a dynamic site equation with polymorphism and variable asymptotes, is to redefine both θ and ζ_1 as functions of an explicit site variable and then replacing this variable with its initial condition solution (Cieszewski 1994). An example of such an approach is the next section. Yet, another approach, that might be simpler in some situations, is to expand the anamorphic site model by adding to it an additional site-independent function of age. For example, adding to the above anamorphic equation the following site-independent function of age:

$$\frac{t\,(\beta_1 \ln t - \gamma_1 t) \ln t}{\zeta_1 + t}$$

results in the expanded model below:

$$H = \frac{t\,\theta_i}{\zeta_1 + t} + \frac{t\,(\beta_1 \ln t - \gamma_1 t) \ln t}{\zeta_1 + t} \tag{5}$$

that is polymorphic and has variable asymptotes—both desired characteristics of many site models (Fig. 1). Next, the appropriate dynamic equation can be derived from the expanded equation by substituting θ_i with its initial condition solution. Thus, since the solution for θ_i is

$$\theta_i = (\gamma_1 t - \beta_1 \ln t) \ln t + H\,(1 + \zeta_1/t) = \gamma_1\, t_0\, \ln t_0 - \beta_1\, \ln^2 t_0 + H_0\, (1 + \zeta_1/t_0)$$

the dynamic site equation can be presented as:

$$H = H_0 \frac{(\zeta_1 + t_0)\,t}{(\zeta_1 + t)\,t_0} + \ln\left(\frac{t_0^{\gamma_1 t_0 - \beta_1 \ln t_0}}{t^{\gamma_1 t - \beta_1 \ln t}}\right)^{\frac{t_0\,t}{\zeta_1 + t}} \tag{6}$$

which happen to be a generalization of the site index model (1) and the height growth model (3). The same dynamic equation may be derived from these two models ((1) and (3)) by applying to them a model constraining technique assuring equality between the site indexes and the heights at base age (Cieszewski 1994).

The forestry literature contains different examples of constraining site index and height growth models to compute heights at base age equal to site index and vice versa. Most approaches focus on substituting parameters with their solutions. The selection of the conditioning coefficients varies. Some practitioners select coefficients adjacent to site index (e.g., Burkhart and Tenant 1977) and others select coefficients remote from it (e.g., Biging 1985). Yet, in other approaches, the equation may be constrained by equating to 1 a ratio of the height at base age and site index (e.g., Curtis *et al.* 1974).

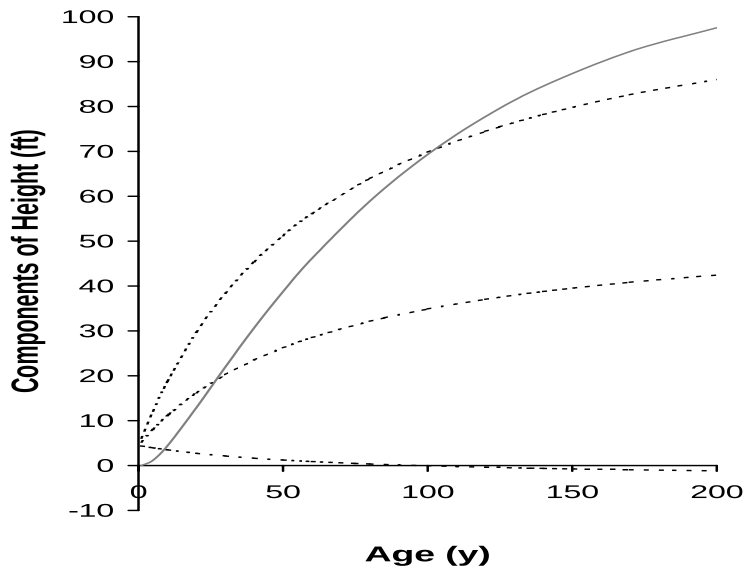

Figure 1: The components of model (6): i) the anamorphic sub-model (thin dashed lines); and ii) the site-independent function of age (thick solid line).

Cieszewski (1994) advocates that the most desirable technique is based on solving site equations for the site index rather than for any of the model parameters. While such an approach is contingent on the availability of site index solutions, it is desirable for at least five reasons:

1) it guarantees that the final model will generate curves identical to those of the initial model;

2) it results in proper, initial-condition difference equations, or dynamic site equations, which are truly base-age invariant;

3) it results in equations which are more flexible and general than the original equations;

4) it identifies and reduces any over-parameterization of the model;

5) it logically resolves what points the curves should pass through at base-ages.

These five advantages are, in principal, always desirable, even if they make only a small difference in operational use. For example, Monserud's (1984) models based on eq. (1) and (2) imply that site index values are very close to heights at base age. The resulting inconsistencies between site indexes and heights at base age are practically negligible. At the same time, it is desirable that these values be equal, while proper constraining will often result in additional benefits, such as, reduction of redundant parameters. This can be well illustrated using the example of eq. (1) and its inverse function (3). To see it, consider that the initial condition solution for S in eq. (3) results from

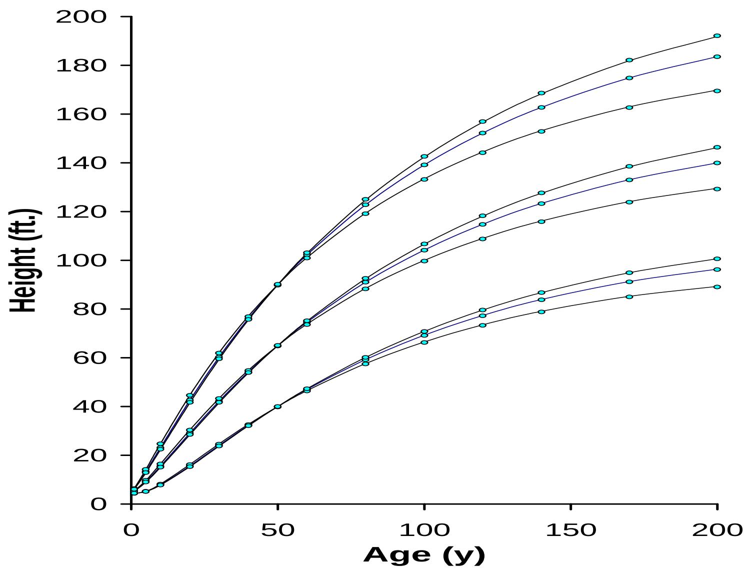

Figure 2: Heights computed by model (3) (symbols) and model (6) (lines). Curves based on the three sets of parameters reported by Monserud (1984) and three productivity sites: 40, 60, and 90 feet.

eq. (1) and it is:

$$S = \alpha - \beta \ln^2 t_0 + \gamma t_0 \ln t_0 + \delta H_0 + \zeta \frac{H_0}{t_0}$$

When the right hand side of this solution is substituted in place of S in eq. (3) the result is the dynamic site equation (6), where: $\beta_1 = \beta/\delta$, $\gamma_1 = \gamma/\delta$, $\zeta_1 = \zeta/\delta$ and H denotes height at age t on a site defined by a height H_0 at an arbitrary age t_0.

The model (6) constitutes a parsimonious, dynamic site equation including the two equations, (1) and (3), as special cases. It is an improvement over equations (1) and (3). It is more parsimonious and more flexible in its applications. It has only three parameters. Yet, without a need for reformulation it can generate all identical predictions (Fig. 2) as both equations (1) and (3). This model is more consistent in its applications than equations (1) and (3). It consistently computes right values of site indexes from heights and ages and right values of heights from ages and site indexes, regardless of the considered productivity site and the data used in fitting of the model.

Furthermore, the dynamic form of eq. (6) allows it to directly predict heights at any age from any other reference height and age, making it more flexible in its application than eq. (1). However, if desired, eq. (6) can be used as a height growth model using a fixed base-age site index if it is, for example, rewritten as:

$$H = S \frac{(\zeta_1 + 50)\, t}{(\zeta_1 + t)\, 50} + \ln \left(\frac{50^{\gamma_1 50 - \beta_1 \ln 50}}{t^{\gamma_1 t - \beta_1 \ln t}} \right)^{\frac{50\, t}{\zeta_1 + t}} \tag{7}$$

It can also be used as a fixed base-age site index model predicting the site index from any height and age if it is rewritten as:

$$S = H\frac{(\zeta_1 + t)\,50}{(\zeta_1 + 50)\,t} + \ln\left(\frac{t^{\gamma_1 t - \beta_1 \ln t}}{50^{\gamma_1 50 - \beta_1 \ln 50}}\right)^{\frac{t\,50}{\zeta_1+50}} \tag{8}$$

A Dynamic Site Equation for the Height Growth Model

The derivation of equations (6)–(8), presented above, preserves all identical properties of both equations (1) and (3). An analogical derivation applied to the height growth model (2), using the approach described by Cieszewski (1994), could be defined as follows. Given the solution, to eq. (2), for the site index:

$$S = u_{(t,H)} = u_{(t_0,H_0)}$$

the dynamic form replacing eq. (2) would be based on

$$H = \frac{\alpha u^{\beta}_{(t_0,H_0)}}{1 + e^{\gamma - \delta \ln t - \zeta \ln u_{(t_0,H_0)}}}$$

However, $u_{(t,H)}$ is very complex and, depending on parameters, it may not be possible to express it with a closed-form equation. Hence, this approach may not always be directly applicable for this type of equation. Conversely, complex nonlinear equations, such as eq. (2), are often excessively flexible, in that they allow for reductions in the number of parameters without significantly restricting their inherent flexibility. Thus, one can explore minor modifications of eq. (2) without negatively impacting its usefulness. For example, I examine different assumptions about the parameters β and ζ. There are several workable alternatives. Some useful options are setting β or ζ to zero or, imposing the restriction of $\beta = \zeta$. Any of these will assure existence of closed-form solutions and reduce the number of parameters. The latter, in particular, does little to limit the flexibility of the new model. At the same time, adding an intercept parameter (η) to the numerator of eq. (2) will increase its flexibility. More parameters can be added. For example, multiplying the right-hand side of the equation by an exponential function of age (t^{v}) further increases the model flexibility. Yet, adding all these parameters does not preclude the closed-form solutions. Thus, a suitably expanded six-parameter model with closed form solutions may be:

$$H = \begin{cases} t^{v}\left(\eta + \alpha S^{\beta}\right)\left(1 + e^{\gamma - \delta \ln t}\right)^{-1} & \text{if: } \zeta = 0 \\ \eta t^{v}\left(1 + e^{\gamma - \delta \ln t - \zeta \ln S}\right)^{-1} & \text{if: } \beta = 0 \\ t^{v}\left(\eta + \alpha S^{\zeta}\right)\left(1 + e^{\gamma - \delta \ln t - \zeta \ln S}\right)^{-1} & \text{if: } \beta = \zeta \end{cases}$$

The above equation is undefined for $t = 0$ and can be simplified with respect to the exponentiated logarithms and therefore, to address these issues, I will rewrite it as a modified (see eq. (4)) Hosfald's

(1822) function:

$$H = \begin{cases} t^{\upsilon+\delta}\left(\eta + \alpha\, S^{\zeta}\right)\left(t^{\delta} + \gamma'\right)^{-1} & \text{if: } \zeta = 0 \\ t^{\upsilon+\delta}\eta\left(t^{\delta} + \gamma'/S^{\zeta}\right)^{-1} & \text{if: } \beta = 0 \\ t^{\upsilon+\delta}\left(\eta + \alpha\, S^{\zeta}\right)\left(t^{\delta} + \gamma'/S^{\zeta}\right)^{-1} & \text{if: } \beta = \zeta \end{cases} \tag{9}$$

where $\gamma' = e^{\gamma}$ and either γ or γ' can be used directly as the estimable parameter.

In other words, the functional form of the complex nonlinear eq. (2) can be modified to assure model flexibility and solvability for the site index. In the end, an appropriate model, such as eq. (9), can be reformulated to a form of a dynamic equation. Such reformulation increases model performance and its flexibility in applications. This also addresses the temporary increase in the number of model parameters. For example, the solution to eq. (9) depends on the values of parameters α and γ' and it can be defined as the following function of the initial conditions:

$$S = \begin{cases} \left.\begin{cases} \left(\left(H_0\left(t_0^{\delta} + \gamma'\right)t_0^{-\upsilon-\delta} - \eta\right)/\alpha\right)^{1/\beta} & \text{where: } \zeta = 0 \\ \left(\gamma' H_0 t_0^{-\delta}\left(\eta' t_0^{\upsilon} - H_0\right)^{-1}\right)^{1/\zeta} & \text{where: } \beta = 0, \\ & \text{and: } \eta' = \eta + \alpha \end{cases}\right\} \text{if } \beta \neq \zeta \\ \left.\begin{cases} \left(\gamma' H_0 t_0^{-\delta}\left(\eta t_0^{\upsilon} - H_0\right)^{-1}\right)^{1/\zeta} & \text{where: } \alpha = 0 \neq \gamma' \\ \left(\left(H_0/t_0{}^{\upsilon} - \eta\right)\alpha^{-1}\right)^{1/\zeta} & \text{where: } \alpha \neq 0 = \gamma' \\ \left(.5\alpha^{-1}\left(\mathcal{Z}_0 \pm \left(\mathcal{Z}_0^2 + 2\kappa H_0/t_0^{\upsilon+\delta}\right)^{.5}\right)\right)^{1/\zeta} & \text{where: } \alpha \neq 0 \neq \gamma', \\ \text{and where: } \quad \mathcal{Z}_0 = \pm H_0/t_0^{\upsilon} \mp \eta \quad \text{and} \quad \kappa = 2\alpha\gamma'. \end{cases}\right\} \text{if: } \beta = \zeta \end{cases}$$

which, in turn, is applied to the six parameter eq. (9) resulting in the following four, or fewer, parameter dynamic equation:

$$H = \begin{cases} \left.\begin{cases} t^{\upsilon+\delta}H_0\left(t_0{}^{-\upsilon} + t_0{}^{-\upsilon-\delta}\gamma'\right)\left(t^{\delta} + \gamma'\right)^{-1} & \text{where: } \zeta = 0 \\ t^{\upsilon+\delta}\eta'\left(t^{\delta} + t_0^{\delta}\left(t_0^{\upsilon}\eta'/H_0 - 1\right)\right)^{-1} & \text{where: } \beta = 0, \\ & \text{and: } \eta' = \eta + \alpha \end{cases}\right\} \text{if } \beta \neq \zeta \\ \left.\begin{cases} t^{\upsilon+\delta}\eta\left(t^{\delta} + t_0^{\delta}\left(t_0^{\upsilon}\eta/H_0 - 1\right)\right)^{-1} & \text{where: } \alpha = 0 \neq \gamma' \\ H_0(t/t_0)^{\upsilon} & \text{where: } \alpha \neq 0 = \gamma' \\ H_0\left(t/t_0\right)^{\upsilon+\delta}\left(t_0^{\delta}\mathcal{R} + \kappa\right)\left(t^{\delta}\mathcal{R} + \kappa\right)^{-1} & \text{where: } \alpha \neq 0 \neq \gamma', \\ \text{and: } \mathcal{R} = \mathcal{Z}_0 + \left(\mathcal{Z}_0^2 + 2\kappa H_0/t_0^{\upsilon+\delta}\right)^{.5} & \text{and: } \mathcal{Z}_0 = H_0/t_0^{\upsilon} - \eta. \end{cases}\right\} \text{if: } \beta = \zeta \end{cases} \tag{10}$$

Note that $\mathcal{Z}_0$ is specified above for the root which, is more likely to be positive and which, was used in this work. For the other root $\mathcal{Z}_0 = \eta - H_0/t_0^{\upsilon}$ and all other symbols are as previously defined. This model has different possible expressions corresponding to limiting cases of the model parameters.

First, if $\zeta = 0$, eq. (10) is a generalization of the anamorphic site equation presented in Amaro *et al.* (1997):

$$H = H_0 \frac{1 + \gamma'/t_0^\delta}{1 + \gamma'/t^\delta}$$

This generalization is equivalent to the equation in Amaro *et al.* (1997) for $\upsilon = 0$, but it has an improved form that is defined for $t = 0$.

Second, if $\beta = 0$, the above equation is a generalization of the polymorphic site equation described in McDill and Amateis (1992):

$$H = \frac{\eta}{1 + \left(\frac{\eta}{H_0} - 1\right)\left(\frac{t_0}{t}\right)^\alpha}$$

Similarly as before, the generalization (10) is equivalent to the above equation if $\upsilon = 0$, but it also has an improved algebraic form that is defined for $t = 0$.

Third, if $0 \neq \beta = \zeta \neq 0$, the dynamic equation can have three additional forms depending on the parameters α and γ'.

If $\alpha = 0 \neq \gamma'$, is equivalent to the case of the assumption $\beta = 0$ described above in the **Second** case.

If $\gamma' = 0 \neq \alpha$, the model is a simple exponential proportionality. However, for $\upsilon = 0$ this case does not define a meaningful relationship.

If $\gamma' \neq 0 \neq \alpha$, the model is polymorphic with variable asymptotes. This case defines probably the most useful and biologically sound relationship. Also, it is a special case generalization of the models applied in: AFS (1985), BCMF (1991), Cieszewski and Bella (1989 and 1993), Elfving and Kiviste (1997), Eriksson *et al.* (1997), Goudie (1984), Kiviste (1997 and 1998), Nigh (1997 and 1998), Nigh and Courtin (1998), Nussbaum (1996), Thrower (1992), Thrower *et al.* (1994), and others. Further simplifying it by the assumption of $\upsilon = 0$ results in the following three-parameter dynamic equation:

$$H = H_0 \frac{t^\delta \left(t_0^\delta \mathcal{R} + \kappa\right)}{t_0^\delta \left(t^\delta \mathcal{R} + \kappa\right)} \tag{11}$$

where

$$\mathcal{R} = \mathcal{Z}_0 + \left(\mathcal{Z}_0^2 + 2\kappa H_0/t_0^\delta\right)^{.5}$$

and $\mathcal{Z}_0 = H_0 - \eta$ for the applied here root that is more likely to be positive. For the other root $\mathcal{Z}_0 = \eta - H_0$ and all the other symbols are as previously defined.

The three-parameter eq. (11) is just one example of several possible cases of eq. (10). However, it is the most generic case and most flexible for many applications. It is an improved form of the equation applied in Cieszewski and Bella (1989 and 1993) and applied in Elfving and Kiviste (1997), Eriksson *et al.* (1997), Kiviste (1997 and 1998) and others, which is:

$$H = \frac{0.5\left(\mathcal{R} + \kappa/\gamma'^{\alpha}\right)}{1 + 2\kappa\left[\left(\mathcal{R} - \kappa/\gamma'^{\alpha}\right)t^{\alpha}\right]^{-1}} \tag{12}$$

where

$$\mathcal{R} = h_0 + \left((h_0 - \kappa/\gamma'^{\alpha})^2 + 4\kappa h_0/t_0^{\alpha}\right)^{.5}$$

and either γ' or $\gamma'' = \gamma'^{\alpha}$ or $\gamma'' = \kappa/\gamma'^{\alpha}$ can be used as the third regression parameter.

Equation (11) is also a generalization of the special case ($\zeta = 1$) of the constrained hybrid of eq. (2) applied in AFS (1985), Nussbaum (1996), Goudie (1984), BCMF (1991), Thrower (1992), Thrower *et al.* (1994), Nigh (1997 and 1998), Nigh and Courtin (1998), and others, which is:

$$H = S\,\frac{1 + e^{\gamma - \delta \ln 50 - \zeta \ln S}}{1 + e^{\gamma - \delta \ln t - \zeta \ln S}} \tag{13}$$

The most obvious advantage of equations (10) and (11) over the equation (13) is its dynamic form using directly heights and ages to predict any heights or site indexes, while eq. (13) does not have a known compatible site index solution. Also, unlike eqs. (12) and (13), the new eqs. (10) and (11) are defined at age equal zero and they allow to readily see that for all base-ages the equations compute heights equal site indexes. Equations (11) and (13) are similar to each other and to eq. (2). They are equivalent to each other for $\zeta \approx 1$ and $\eta \approx .5\kappa/50^{\delta}$. When $\zeta \approx 1$ and η is a free parameter, eq. (11) is a generalization of eq. (13). When $\eta \approx .5\kappa/50^{\delta}$ and ζ is a free parameter then eq. (13) is a limited case generalization of eq. (11). The limited case of this generalization relates to the fact that eq. (13) is operable only for one base age 50 years. Equation (11) is operable for all possible base ages and therefore, cannot be generalized by eq. (13). Finally, eq. (11) with $\eta \approx 0$ is equivalent to eq. (2) with its $\zeta \approx \beta$ while eq. (11) with η as free parameter is a generalization of eq. (2) with $\zeta \approx \beta$.

The four-parameter eq. (10) is very flexible and, for practical purposes, it may be considered as more flexible than the five-parameter eq. (2). With many data sets the three-parameter eq. (11) will be as flexible as the five-parameter eq. (2). In other cases eq. (11) may be just flexible enough to justify dropping nearly half of the parameters in eq. (2) and gaining more direct and consistent model applications. For example, I have compared curves generated by eq. (11) with curves generated by eq. (2) using the parameter values from Monserud (1984). The curves generated by eq. (11) using directly γ, δ and ζ from eq. (2) and ignoring completely the parameters α and β were very similar

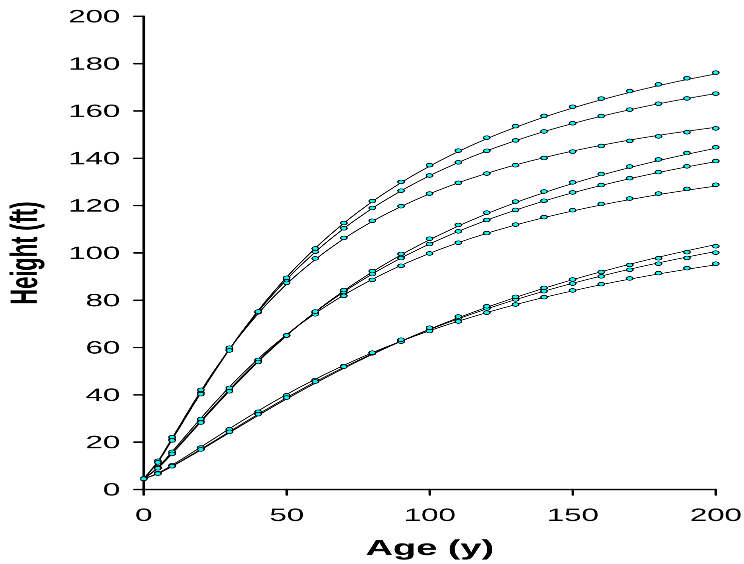

Figure 3: Heights computed by model (2) (symbols) and model (11) (lines).

to curves generated by eq. (2). When the parameters κ, δ and ζ, in eq. (11), were fitted to the predictions of eq. (2), the curves generated by eq. (11) were virtually identical to the ones generated by eq. (2). The degree of similarity between these curves is illustrated on Figure (3).

Equations (2) and (11) assume that a half-saturation times are inversely proportional to an exponential function of the site productivity. Equation (2) assumes that the maximum yields are directly proportional to exponential function of site indexes. Equation (11) assumes that these yields are linearly proportional to an exponential function some kind of site variable but not necessarily site indexes. This means that the maximum yields could be, for example, proportional to the ozone level, that is, inversely proportional to site indexes. Also, due to the linear intercept, the later assumption is more flexible and seems to be biologically more relevant because, it allows for a more precise definition of the asymptote response space. For example, one special case of the model can have a minimum response of asymptotes in upper range of their values, while another case can have asymptotes directly proportional to site indexes in all range of values as in eq. (2).

For some applications the three-parameter eq. (11) may not be sufficiently flexible. For example, in situations of stand volume modeling the required trends may need to decline at older ages with stand decadence. Models of such trends are required in timber supply analysis, annual allowable cut calculation and other analysis of forest sustainability and estate management. In such situations, eq. (10) might be more suitable than eq. (11).

Finally, the eq. (11) can be also written as a static fixed base age site index equation and a similar height growth equation. For the base age 50 the corresponding site index equation can be written as:

$$S = H \frac{50^{\delta}\left(t^{\delta}\mathcal{R} + \kappa\right)}{t^{\delta}\left(50^{\delta}\mathcal{R} + \kappa\right)} \tag{14}$$

where $\mathcal{R} = \mathcal{Z} + \left(\mathcal{Z}^2 + 2\kappa H/t^{\delta}\right)^{.5}$ and $\mathcal{Z} = H - \eta$. A similar fixed base age height growth equation can be written as:

$$H = S \frac{t^{\delta}\left(50^{\delta}\mathcal{R} + \kappa\right)}{50^{\delta}\left(t^{\delta}\mathcal{R} + \kappa\right)} \tag{15}$$

where $\mathcal{R} = \mathcal{Z} + \left(\mathcal{Z}^2 + 2\kappa S/50^{\delta}\right)^{.5}$ and $\mathcal{Z} = S - \eta$.

Discussion

The new dynamic equations have the flexibility to use, instead of site indexes, directly heights at any age and to generate site index and height curves. Each of them has only three parameters. Yet, they generate curves that are identical or almost identical to those generated by the original five-parameter equations.

Monserud (1984) expresses a consideration that constraining of site models may create an undue importance to the index age. Such importance is given by selection of a statistical approach to fitting the height prediction system. Specifically, the fitting of the site index models with observed heights used as site indexes creates the dependence between the base-age and the estimates of model parameters. Examples of alternative approaches are in: Bailey and Clutter (1974), Begin and Schutz (1994), Duplat and Tran-Ha (1986 and 1997), Garcia (1983), and Tait et al. (1988) . In retrospect, one can include in the model an explicit measure of the deviation of site indexes from the measured heights at base age. This may require a use of "nuisance parameters" temporary redefining site index as a product, sum, or other function of this deviation and the site index. For example, eq. (6) can be fitted applying a temporary (in the fitting only) "nuisance parameter" θ_i describing the site index deviation from the observed in the data values:

$$H \leftarrow (\theta_i + S)\frac{(\zeta_1 + 50)\, t}{(\zeta_1 + t)\, 50} + \ln\left(\frac{50^{\gamma_1 50 - \kappa_1 \ln 50}}{t^{\gamma_1 t - \kappa_1 \ln t}}\right)^{\frac{50\, t}{\zeta_1 + t}} \tag{16}$$

where: "$\leftarrow$" denotes one way assignment (as opposed to equality), θ_i is the "nuisance parameter" and all other symbols are as defined earlier. "$\leftarrow$" signifies that θ_i voids the equality, while temporarily used in the regression. θ_i can be a single parameter or a vector of site specific parameters for a base age invariant parameter estimation similar to the methods discussed in Bailey and Clutter (1974), Begin and Schutz (1994), Duplat and Tran-Ha (1986 and 1997), Garcia (1983), and Tait et al. (1988)

, as well as, Linstrom and Bates (1990). In the first case, θ_i describes an average site index deviation from the heights at base age similarly as *alpha* in eq. (1). In the second case, θ_i describes individual deviations for each growth series. The nuisance parameter is discarded after the model fitting in a similar way as the autocorrelation coefficients are. An example of using a "nuisance parameter" in Monserud (1984) is ϕ in model 4a that should be defined as: $Y_n = f(X_n, \boldsymbol{\beta}) + \phi\epsilon_{n-1} + \nu_n$, where ϵ_{n-1} is the regression lagged residual and ν_n is the minimized stochastic element. Even though ϕ is estimated during the model fitting, it is not used in the model applications.

The explicit equations, such as eq. (5), could be fitted directly to height and age data but implicit equations, such as, eq. (6) are likely to behave better and give more flexibility in model fitting and its applications. The earlier equation is a good example of how one can extend the flexibility of an explicit site model without precluding its solvability for the site variable. Such technique is a powerful tool in developing base-age invariant dynamic equations. It enables a relatively easy derivation of polymorphic dynamic site equations with variable asymptotes—the algebraically most difficult and scarce class of base-age invariant dynamic site equations. Another example of a similar technique can be found in Duplat and TranHa (1997).

The derivation of eq. (10) illustrates an example of the methodology described in Cieszewski (1994). This methodology has been designed for derivation of biologically based base-age invariant dynamic site equations. It basically consists of using a theoretical site variable to formulate biological theories about different characteristics of growth dynamics. Such characteristics can include the limiting size and the half-saturation time as in the presented here example of eq. (9). The final model is derived from substitution of the theoretical site variable with its initial condition solution.

The dynamic equations are likely to be more flexible and parsimonious than their corresponding fixed base age site index and height growth equations. They can be derived using solutions for site index without or with modifications to the original base models. If needed, the modifications, just as the base models, can use sums of functions with and without site responses. The only situations justifying development of a fixed base age site index or height growth models are those in which comparable dynamic equations cannot be derived. However, this does not apply to such models as, for example, the five-parameter site index eq. (1). Once the three-parameter dynamic equation (6) has been identified, it would be, from both statistical and practical points of view, undesirable to use eq. (1), which is less parsimonious, less flexible, and violates the assumption on equality between site index and height at base-age.

Lastly, if for any reasons modelers need to work with fixed base age site index models they will benefit from testing their models against corresponding dynamic equations. These will provide at least a practical check of parsimony for the fixed base-age equations. One can also assume a fixed base-age property of any dynamic equation by simply replacing the variable base-age (t_0) with an arbitrary age, such as, 50, as illustrated in the equations (7), (8), (14), and (15).

Summary and Conclusions

Equations (1) and (2) proved very popular for forestry applications. They have a number of desirable characteristics:

- they are polymorphic;
- have variable asymptotes; and
- fit many data sets well.

I have presented here improved versions of these equations with additional desirable characteristics. These include:

- heights at base ages equal site indexes;
- dynamic equation forms with direct use of height/age data;
- common equation forms for site index and height computations;
- defined origin at time equal zero;
- base-age invariance;
- increased parsimony; and
- expanded flexibility.

The new dynamic equations are recommended in place of equations (1) and (2). The methods presented are recommended for all applications requiring the use of implicitly defined equations in modeling all such characteristics as: height, diameter, basal area, volume, biomass, carbon, and trees per unit area.

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