Example of Mathematically Adapting Fixed-Base-Age Models to Variable-Base-Age Inventory Projections

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Abstract

The generalized log-logistic height equation computes height as a function of age and a fixed base-age site index. This equation and its modifications have been used for many applications in various regions. It is seemingly apparent that this equation is analytically insolvable for the site index; no general analytical solution to this equation is readily available. This article presents such a solution that has proven to be valid and useful with all tested parameters. This solution is based on an adaptation of the Ramanujan's (1887-1920) solution for a trinomial with real-number exponents. Ramanujan's solution is a series that in many cases can be expressed as a closed-form equation. In the present context, this series may be used for derivations of various special-cases of closed-form solutions, initial-condition difference and differential equations, and for various analytical sensitivity and trend analysis, as well as, for efficient site index computations.

Keywords: analytical solutions, numerical solutions, polynomial solutions, site index models, base-age invariant, convergent series.

Background

In this article the term "site index equation" denotes an equation defining site index (S) as a function of height (H) and age (t). An equation defining H as a function of S and t is termed "height equation". An inverse function of a height equation is its compatible site index equation; it is the site index solution to the height equation. The subject of this article is the site index solution to the following height equation (Monserud 1984):

$$H = \frac{\alpha S^{\beta}}{1 + e^{\gamma - \delta \ln t - \zeta \ln S}} \tag{1}$$

where α, β, γ, δ, and ζ are the model parameters and all other symbols are as defined above.

To apply this equation one needs to know S, which is dependent on an available H measured at any age t. Yet, no compatible site index equation is available for eq. (1). My objective is to derive such an equation; I pursue the analytical solution for S in eq. (1). Such a solution is the mathematical inverse function of eq. (1), defining analytically S in the form of the site index equation compatible with eq. (1), rather than any of its specific values, which could be and at times have been, approximated numerically.

The Analytical Solution to a Trinomial with Real-exponents

The most straightforward derivation of an analytical solution is through manipulation of mathematical symbols. Even when this is not possible, some equations can still be analytically solved through formulas for their roots. The simplest example is the quadratic equation, $\alpha + \beta X + \gamma X^2 = 0$, that cannot be manipulated to isolate X but can be analytically solved for X using appropriate root equations.

Polynomials of degree 3 and 4 are also solvable for their roots by means of standard formulas, but higher degree general polynomials are, by definition, proven by Galois[1], insolvable. On the other hand, there are many special-case polynomials of higher degrees that are analytically solvable. For example, the n-degree polynomial, $\alpha + \beta X^{n/2} + \gamma X^n = 0$, can be solved as a quadratic equation with a substitution $\mathcal{X} = X^n$.

Polynomials with real-exponents are particularly difficult to solve. They are generally insolvable due to a potentially infinite number of complex roots. Nonetheless, in the course of history, mathematicians have identified some special-cases of seemingly insolvable polynomials that are analytically tractable and can have <u>exact</u> solutions under specified conditions. Without presenting all the proofs

[1] Évariste Galois (1811–1832): French mathematician with main contributions in number theory.

that can be found in the source, I explore here one example (Ramanujan 1985) of such a solution to a seemingly insolvable real-exponent polynomial that is key to obtaining the analytical site index solution to, or inverse function of, eq. (1).

Ramanujan (1985) considers a special-case real-exponent polynomial in the form of the following trinomial

$$aqx^p - x^q + 1 = 0 \tag{2}$$

where p and q are real numbers. This trinomial has the following analytical solution:

$$x = (\varphi(n))^{1/n} \tag{3}$$

where for every real number n, x is the root of the trinomial (2) and φ is defined as

$$\varphi(n) = \sum_{k=0}^{\infty} \frac{c_k(n)a^k}{k!} \tag{4}$$

where: $c_0(n) = 1$, $c_1(n) = n$ and

$$c_k(n) = n \prod_{j=1}^{k-1} (1 + kp - jq) \ , \qquad \text{for:} \qquad k \geq 2.$$

It is clear that the solution in $\varphi(n)$ is convergent when $|a| < 1$. Such a convergent series represents a finite number and constitutes the exact solution to the trinomial (2) despite its seemingly apparent aspect of infinite summations. Furthermore, Ramanujan (1985) proves that this solution is convergent for all

$$|a| \leq p^{-p/q}|p - q|^{(p-q)/q} \tag{5}$$

in which case, the series may be convergent even if $|a| \geq 1$; the solution is exact and the series can be used to calculate the solution's exact values for any given set of parameters. I will explore this fact further in defining the solution for eq. (1), but first, I discuss the interpretation of the above series solution.

Interpretation of the Analytical Solution

I demonstrate the exactness of the solution (4) using various assumptions about the parameters p and q. Initially I consider the simplest case with $p = q = 1$. With this assumption the trinomial (2) becomes

$$aqx^p - x^q + 1 = ax - x + 1 = -x(1 - a) + 1 = 0 \tag{6}$$

which can be trivially reformulated to the exact closed-form solution for x:

$$x = \frac{1}{1-a} \tag{7}$$

I further compare this solution with the one obtained from the infinite but convergent series (4).

Before applying the solution in φ defined by the eq. (4), to compute the root ($x = \varphi(1)$) for eq. (6), I choose $n = 1$ for clarity of presentation and note that if $k = 0$ then $c_0(1) = 1 = 0!$, if $k = 1$ then $c_1(1) = 1 = 1!$, and if $k \geq 2$ then $c_k(1)$ is defined as

$$\begin{aligned}
c_2(1) &= 1 + 2 - 1 = 2 = 2! \\
c_3(1) &= (1 + 3 - 1)(1 + 3 - 2) = 3 \times 2 = 3! \\
&\vdots \\
c_k(1) &= (1 + k - 1)(1 + k - 2)\ldots(1 + k - (k-2))(1 + k - (k-1)) = k!
\end{aligned}$$

Defining $c_k(1)$ as $k!$ simplifies the following derivation when further calculating the root of eq. (6) defined by the series (4), because $x = \varphi(1) = \sum_{k=0}^{\infty}\left(c_k(1)a^k/k!\right) = \sum_{k=0}^{\infty}\left(k!a^k/k!\right) = \sum_{k=0}^{\infty} a^k$ and therefore:

$$\begin{aligned}
x &= \textstyle\sum_{k=0}^{\infty} a^k &&= 1 + a + \sum_{k=2}^{\infty} a^k \qquad (8) \\
x - 1 &= \textstyle\sum_{k=1}^{\infty} a^k &&= a + \sum_{k=2}^{\infty} a^k \\
\text{for } a < 1,\ a^{\infty} \equiv 0, \text{ and: } \quad \frac{a + a^2 + \ldots + a^{\infty}}{a} &= 1 + a + \ldots + a^{\infty-1} &&= 1 + a + \ldots + a^{\infty} \\
\text{therefore: } \quad \frac{x-1}{a} &= \textstyle\sum_{k=0}^{\infty} a^k &&= 1 + \sum_{k=1}^{\infty} a^k \\
\text{and from eq. (8): } \quad \frac{x-1}{a} &= x \quad \text{only if:} \quad x &&= \frac{1}{1-a}
\end{aligned}$$

which means that for the given specifications

$$\varphi(1) = \sum_{k=0}^{\infty} \frac{c_k(1)a^k}{k!} = \frac{1}{1-a} \quad \text{for} \quad p = q = n = 1 \tag{9}$$

Since the final solution in eq. (9) is identical to the earlier solution in eq. (7) the above example clearly demonstrates that φ defined by eq. (4) provides an exact solution to the discussed trinomial ($ax - x + 1$). If the values of p and q are not restricted to equal 1, as in the above example, but merely equal to each other and different from zero (i.e., $p = q \neq 0$), the general closed-form solution has the following form:

$$\varphi(n) = \sum_{k=0}^{\infty} \frac{c_k(n)a^k}{k!} = \frac{1}{(1-ap)^{n/p}} \qquad \text{for:} \qquad p = q \neq 0 \tag{10}$$

from which the root (x), as per eq. (3), is

$$(\varphi(n))^{1/n} = \left(\sum_{k=0}^{\infty} \frac{c_k(n)a^k}{k!}\right)^{1/n} = x = \frac{1}{(1-ap)^{1/p}} \qquad \text{for all:} \qquad p = q \neq 0 \tag{11}$$

which is easily derived analytically by a simple manipulation of the trinomial.

Similarly, other simplified cases can be derived from various assumptions of the parameters' values, such as, for example, if $p = 0$ and $q \neq 0$, then

$$(\varphi(n))^{1/n} = \left(\sum_{k=0}^{\infty} \frac{c_k(n)a^k}{k!}\right)^{1/n} = x = (1+aq)^{n/q} \quad \text{for:} \quad p = 0 \text{ and } q \neq 0 \tag{12}$$

or, if $p = 2q$, the root of eq. (2), which tends to unity as a tends to 0 is given by

$$x = \varphi(1) = \left(\frac{1-\sqrt{1-4aq}}{2aq}\right)^{1/q} = \left(\frac{2}{1+\sqrt{1-4aq}}\right)^{1/q} \tag{13}$$

which can be verified with the standard quadratic root equations. In a similar vein, if $q = 2p$, the root of the trinomial, which also tends to 1 as a approaches 0, is given by

$$x = \varphi(1) = \left(ap + \sqrt{a^2p^2+1}\right)^{1/p} \tag{14}$$

Other closed form solutions will depend on specific values of the parameters p and q. Yet, the given examples demonstrate the finite and exact aspect of the solution defined by a convergent series. An infinite series that is convergent represents an exact finite number that is calculated as its limit. This limit is equal to the sum of all the series terms.

The Analytical Site Index Solution to the Height Model and its Applications

The height equation (1) was formulated through modifications of the logistic equation (MacKinney *et al.* 1937) and therefore, it is referred to as the modified logistic equation or generalized logistic equation. However, since $e^{\ln X} = X$ for any X, this equation can be simplified to a rational function:

$$H = \frac{\alpha\, S^{\beta}}{1+e^{\gamma-\delta\,\ln(t)-\zeta\,\ln(S)}} = \frac{\alpha\, S^{\beta}}{1+\eta t^{-\delta}S^{-\zeta}} = \frac{\alpha\, S^{\beta}}{1+\frac{\eta}{t^{\delta}S^{\zeta}}} \tag{15}$$

where $\eta = e^{\gamma}$ and either e^{γ} or η can be used depending on data and fitting preferences.

Solving eq. (15) for S is equivalent to solving eq. (1) and it involves finding roots of the following real-exponent polynomial:

$$Ht^{\delta}S^{\zeta} - \alpha t^{\delta}S^{\beta+\zeta} + \eta H = 0 \tag{16}$$

where all symbols are as defined earlier and β and ζ are real numbers.

I note that polynomial (16) is just a trinomial, (as eq. (2)) and therefore, I modify it with several substitutions to re-scale it to a similar form to eq. (2), which has the presented analytical solution. Assuming $\mathcal{A} = t^\delta/\eta$, $\mathcal{B} = \mathcal{A}\,\alpha/H$, $q = \zeta + \beta$, and $p = \zeta$, the trinomial (16) can be written as

$$\mathcal{A}\,S^p - \mathcal{B}S^q + 1 = 0 \tag{17}$$

The assumption of $\mathcal{B}S^q = x^q$ implies that $x = \sqrt[q]{\mathcal{B}}S = \mathcal{B}^{1/q}S$ and $S = \mathcal{B}^{-1/q}x$, and it results in:

$$\mathcal{A}\left(\mathcal{B}^{-1/q}x\right)^p - \mathcal{B}\left(\mathcal{B}^{-1/q}x\right)^q + 1 = \mathcal{A}\mathcal{B}^{-p/q}x^p - x^q + 1 = 0$$

Next, defining $a = \mathcal{A}\,/\,\left(\mathcal{B}^{p/q}q\right)$, and substituting it into the above equation results in the Ramanujan's trinomial (2) with an analytical solution for x given by eq. (4).

Finally, since $\mathcal{B}^{1/q}S = x$, the analytical solution for the site index in eq. (1) is

$$S = \left(1 + a + \sum_{k=2}^{\infty} \frac{a^k \prod_{j=1}^{k-1}(1 + \zeta k - jq)}{k!}\right)\mathcal{B}^{-1/q} \tag{18}$$

where: $\mathcal{B} = \alpha t^\delta/\left(\eta H\right)$, $q = \zeta + \beta$, and $a = t^\delta/\left(\eta\mathcal{B}^{\zeta/q}q\right)$.

Equation (18) concludes the derivation of the analytical solution for site index (S) in eq. (1). The convergence of this solution depends on the values of a. These values, in turn, depend on the parameters of eq. (1) and the input age and height of eq. (18); in essence, the convergence of this solution depends on input ages and site productivity, or site index. For the parameters of the "general model" reported in Monserud (1984), the solution (18) converges, and therefore, is exact whenever $|a| \leq 1.34$. Furthermore, the time of convergence is inversely proportional to the value of a; the smaller a is, the faster is the convergence. Figure 1 illustrates the values of a as a function of input ages and site indexes for the same model. Except for one combination of extreme values of age and site index, all values of a are below 1 and even the extreme value of a is still much smaller than the limit of the solution applicability (1.34). As illustrated on Figure 2, the limit of the solution validity is well within the bounds of practical consideration for all input ages and productivity sites.

Depending on specific parameter values of eq. (1), the series (18) can imply different closed-form equations. For example, one can assume that $\beta = \zeta = 1$ similarly as in Cieszewski and Bella (1989 and 1993), Elfving and Kiviste (1997), Eriksson *et al.* (1997), Kiviste (1997 and 1998), Nigh (1997), and others. Then, the site index solution to eq. (1) is

$$S = \frac{H \pm \sqrt{H\left(H + 4\,\alpha\,e^\gamma t^{-\delta}\right)}}{2\alpha} \tag{19}$$

and the implied initial-condition difference equation, or the base age invariant dynamic site equation,

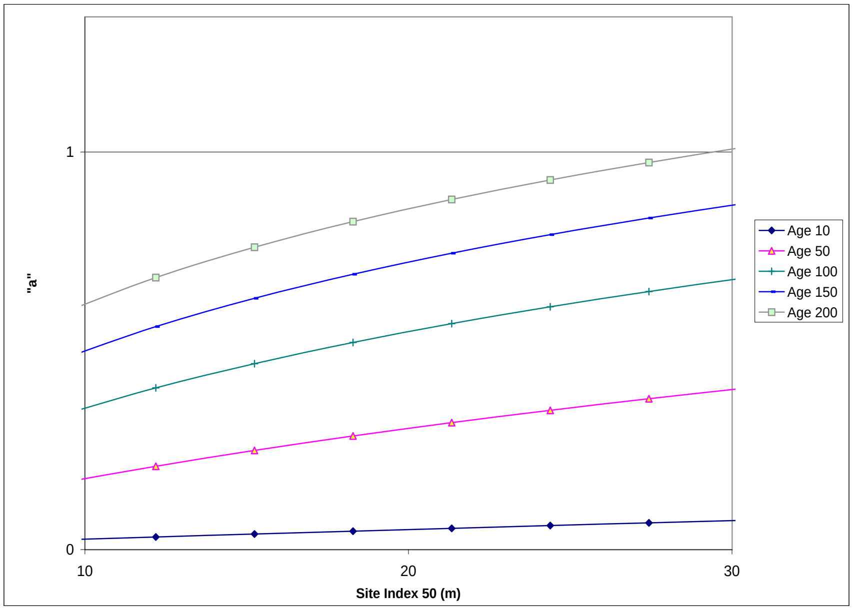

Figure 1: Site and age-dependent values of a for the Monserud (1984) parameters of the general model (eq. (1)). Exact site index solution exist for $a < 1.34$. The value of a is non-dimensional, and the site index base-age is 50 years.

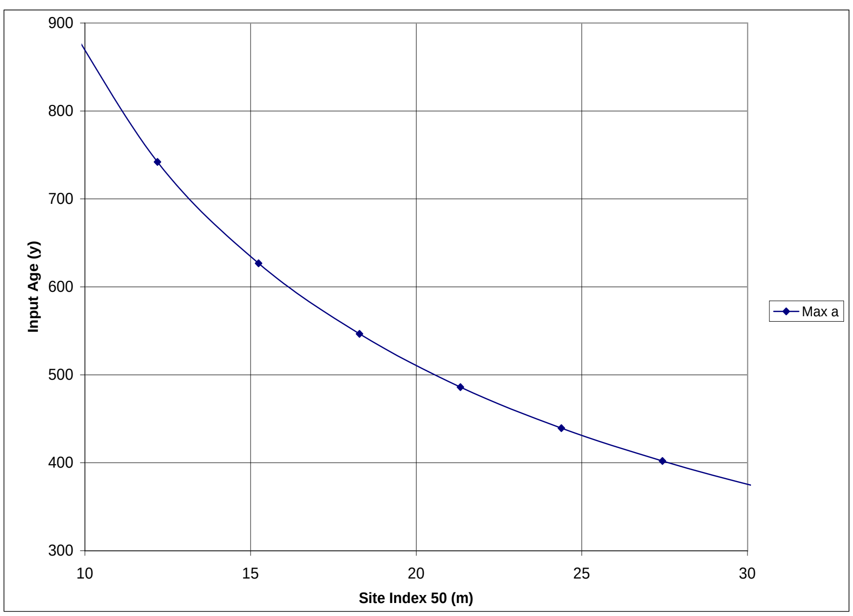

Figure 2: Trinomial solution domain in terms of age and site index. Combinations of site indexes and current age below the full-line provides an exact site-index solution ($a < 1.34$). The value of a is non-dimensional, and the site index base-age is 50 years.

is

$$H = \frac{.5\left(H_0 + \sqrt{H_0\ \left(H_0 + 4\,\alpha\, e^{\gamma} {t_0}^{-\delta}\right)}\right)^2}{H_0 + \sqrt{H_0\ \left(H_0 + 4\,\alpha\, e^{\gamma} {t_0}^{-\delta}\right)} + 2\,\alpha\, e^{\gamma} t^{-\delta}} \tag{20}$$

Furthermore, given the derivative of eq. (1) with respect to t

$$H' = \frac{\alpha\, S^2 e^{\gamma} t^{-\delta} \delta}{\left(S + e^{\gamma} t^{-\delta}\right)^2 t}$$

and the solution (19), the differential equation governing growth of eq. (1) is then:

$$\frac{\partial H}{\partial t} = \frac{\left(H_0 + \sqrt{H_0\ \left(H_0 + 4\,\alpha\, e^{\gamma} {t_0}^{-\delta}\right)}\right)^2 \alpha\, e^{\gamma} t^{-\delta-1} \delta}{\left(H_0 + \sqrt{H_0\ \left(H_0 + 4\,\alpha\, e^{\gamma} {t_0}^{-\delta}\right)} + 2\,\alpha\, e^{\gamma} t^{-\delta}\right)^2} \tag{21}$$

There are many other possible assumptions that would result in various closed-form solutions, initial-condition difference, and differential equations, which are outside of the scope of this article.

Discussion

Series (18) fulfils the objective of this article defining the analytical solution to eq. (1). It is an analytical solution in a form of an inverse factorial series. Within specified assumptions, this series is convergent and defines an exact solution for site index in eq. (1). For many values of the parameters β and ζ the series (18) can be written as a closed-form equation. Even in situations where the closed-form of this solution is unknown, the application of the series should not discourage the consideration of this solution as inverse factorial series generally converge fast.

This means that although, the presented analytical solution can be proven to have an exact value, as, by definition, every convergent series can, its values in practical applications may need to be approximated by a limited number of terms in the series. In such cases, the terms of the series are added until the value of the sum does not change appreciably. Computer codes for such applications in FORTRAN, C, Maple[2] V, Mathematica[3], and Excel are available on request from the author.

Depending on the value of a, convergence may take different numbers of summations. The smaller a is, the faster the convergence. Since a depends on the input age and height, and thus productivity sites, or site index, (Fig. 1), the convergence may require various numbers of summations for different inputs. Solutions for low productivity sites and young ages converge faster. For example, in such a case, a six-digit precision may require just a few summations (Fig. 3a). A similar solution should require only a several summations for medium productivity sites (Fig. 3b), and a dozen for higher

[2] Maple and Maple V are registered trademarks of Waterloo Maple Inc.

[3] Mathematica is registered trademark of Wolfram Research, Inc.

productivity sites (Fig. 3c). When truncating the series summations at a certain fixed number of terms, computations precision will depend on the input age and height, or productivity site. The biggest age-dependent error occurs at about 150 years for high sites, and about 200 years for medium sites. For height-dependent errors, the corresponding values are 55 and 45 m. Errors for a fixed number of terms decrease with decreasing ages and site productivity.

Frequently, when the equation parameters are known, a specific solution may also be approximated through the application of generic numerical searches. Of course, numerical searches are not alternatives to analytical solutions but merely single-use substitutes for them. When searching for a numerical solution, it is incumbent upon the users to assure that any obtained number represents a global solution and is not based on a false convergence or a local minimum. This applies to numerical searches in general; it does not apply to analytical solutions within the defined domain of their applicability.

The understanding of the distinction between analytical and numerical solutions is important in order to appreciate the trinomial analytical solution in the context of height growth modeling. The discussed here framework can be defined as follows. First, a solution here means a root of a polynomial ("solution" and "root" of a polynomial are synonyms). An analytical solution here is an equation defining symbolically a theoretical rule that satisfies a considered relationship, that is, a rule that assures that the equation holds equality for any numbers replacing its symbols. Such an analytical solution associates infinite quantity of numbers, which could be real, complex, etc. A numerical solution is a single real number minimizing a sum of calculated deviations for a given single set of parameter values. Thus the meaning of the word "solution" is different for each of these cases. The analytical solution means a rule/law and the numerical solution means a number/value. The former associates a recipe and the later an algorithm. Finally, the former connotes knowledge/understanding and the later example/case; they subsequently answer the questions why and how versus what and how much.

The provided analytical solution (18) can be used for obtaining compatible site index solutions to different height equations based on eq. (1) whenever numerical searches are applicable. On the other hand, comparisons made in Maple V (Release 5) suggested that solution (18) was a more attractive alternative because the program actually failed to find the roots numerically.

The analytical solution is primarily an analytical tool. It can be used to produce a number if required, but more importantly, it can be useful for much broader analytical work. For example, studying different general properties of the solutions and their dynamics without assuming specific values for the model parameters. Other uses are in derivations of dynamic equations, derivatives,

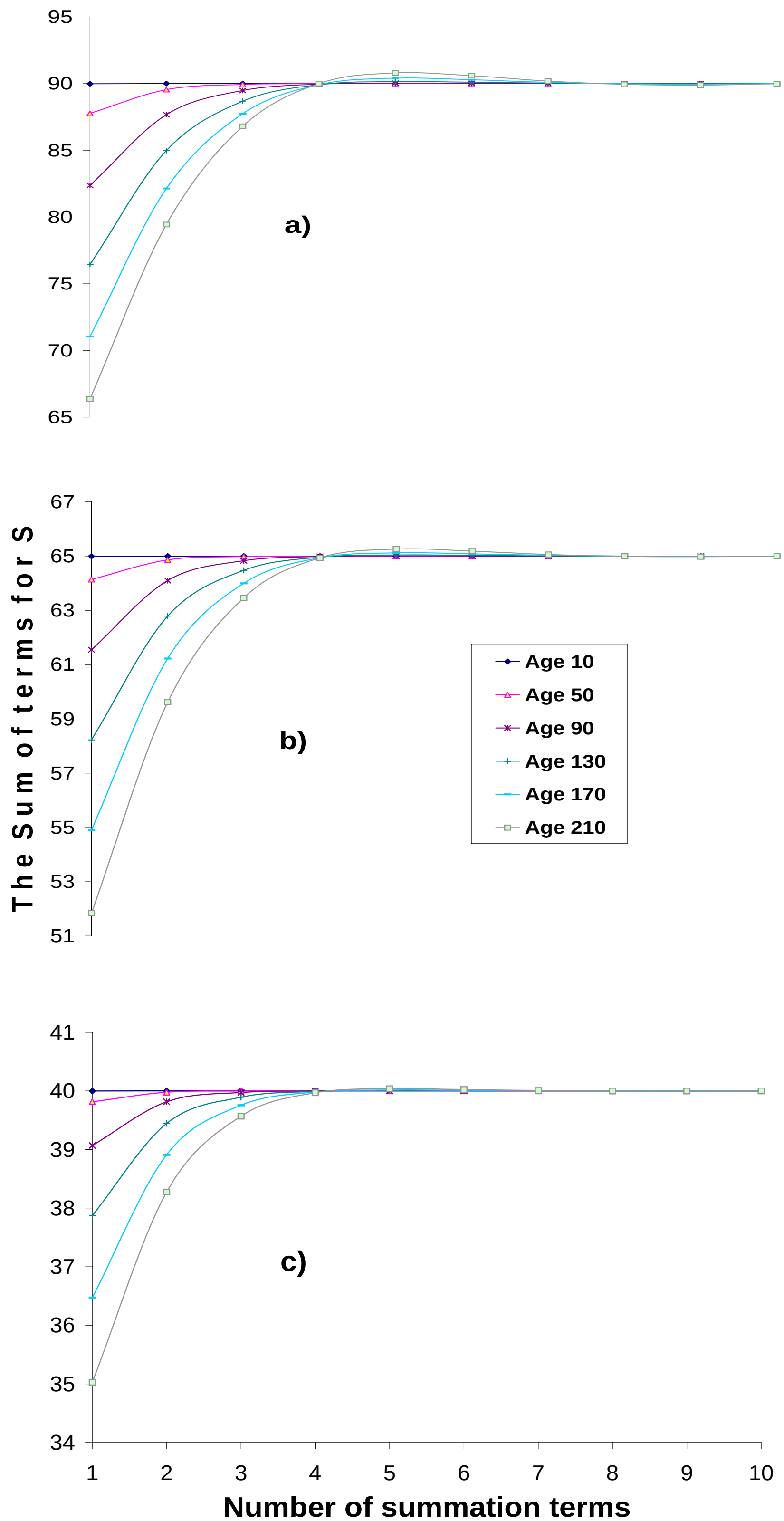

Figure 3: Convergence of 10 summation terms (horizontal axis) for the series solution to the general model in Monserud (1984). The site indexes are in the original imperial units to illustrate the actual solution values: a) high site ($S = 90$ feet); b) medium site ($S = 65$ feet); and c) low site ($S = 40$ feet).

and differential equations describing symbolically growth dynamics for different assumptions of the model exponents. The presented closed-form solutions are just a few examples of such uses. Many other assumptions can be explored each with their own new closed-form solutions.

Various, limited-case, closed-form solutions can be used for sensitivity analysis, for the development of new applications and model forms, and for the study of the properties of growth and yield models. A different practical approach to obtaining a closed-form site index solution compatible with eq. (1) may be using explored by dynamic equations based on analytical site index solutions to height equations, as illustrated by equation (20). A practical demonstration of such work is the base age invariant site equation described in Cieszewski and Bella (1989) and further applied in Cieszewski and Bella (1993), Eriksson *et al.* (1997), Elfving and Kiviste (1997), and Kiviste (1997 and 1998). Especially when the coefficient ζ in eq. (1) is equal to 1, this base age invariant site equation should produce predictions very similar to those from eq. (1) yet still provide compatible closed-form site index solutions.

Finally, the presented solution is also a framework for exact and unbiased analysis and comparison of regression models for height versus site index predictions. Monserud (1984) compared two equation forms with distinctly different properties. This means that some differences between the presented trends originated from the data, but some could have come from functional disparities between the two equation forms. Double-checking of those differences with eq. (1) and its compatible inverse function (4) could produce insights and opportunities in testing their real significance.

End Note

The source of the solution presented here is a Ramanujan's special-case trinomial derivation contained in "*Ramanujan's Notebooks*" (Ramanujan 1985). Ramanujan Srinavasa Ayengar was born On December 22, 1887 near Kumbakonam, Madras (now Tamil Nadu) a town in Southern India. Living in poverty and lacking University education he taught himself mathematics from just two textbooks, borrowed at ages 12 and 15. The later textbook, *A Synopsis of Elementary Results in Pure and Applied Mathematics* by G.S. Carr, published in 1880, became the basis of all his work while he became perhaps the greatest contemporary mathematician in function theory and number theory.

Ramanujan obsessively studied mathematics on his own and was able to discover many existing and new theorems that he was scrupulously recording in his notebooks. In December 1903, he attempted to study at the University of Madras, but at the end of his first year he failed English and physiology. Four years later he tried and failed again. In 1910 he was awarded a monthly stipend from R. Ramachandra Rao, a relatively wealthy mathematician, who already then was indelibly impressed with the contents of Ramanujan's notebooks.

Ramanujan's fortune turned around on February 9, 1912 when, to earn a living, he took a clerical position in the Madras Port Trust Office, whose both the chairman and the manager took a great interest in his welfare and encouraged his contacts with English mathematicians. After some rejections and being taken for a "crank", Ramanujan managed to make acquaintance with G.H. Hardy who invited him to Cambridge, England. At the same time, the University of Madras offered Ramanujan a scholarship starting May 1, 1913—the first official recognition. Later, from 1914 to 1919 he worked at Cambridge with Hardy who published Ramanujan's collected papers in 1927—seven years after the 32-year old author died.

Ramanujan had many great accomplishments, but he is considered a genius mostly for his inexplicable ability in the handling of series and continued fractions. "Ramanujan's Notebooks" were originated during his school years prior to 1903 and were completed before his departure for England in 1914. They were published only in 1957 in an unedited, exact photostat version of the handwritten notebooks, with all the "scratch work", errors, and unfinished parts. An edited version was not published until 1984. The notebooks contain a wide variety of advanced mathematical works including the solution presented here, which Ramanujan discovered independently not knowing that others had published similar works.

According to Ramanujan (1985), solutions similar to the one presented here have a long history. A similar solution was first established by Lambert[4] in a paper published in 1758. In 1770 Lagrange[5] also derived a similar solution, as an application of the "Lagrange inversion formula". Euler[6] published in 1779 a proof of such a formula in a paper stipulated by the work of Lambert. Many others have published other closely related studies.

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[4] J.H. Lambert (1728–1777), a German mathematician, physicist, and philosopher.

[5] J.L. de Lagrange (1736–1813), a French mathematician and astronomer.

[6] L. Euler (1707-1783), a Swiss mathematician.

Literature Cited

Cieszewski, C.J., and I.E. Bella. 1989. Polymorphic height and site index curves for lodgepole pine in Alberta. Can. J. For. Res. 19: 1151–1160.

Cieszewski, C.J., and I.E. Bella. 1993. Modeling density-related lodgepole pine height growth, using Czarnowski's stand dynamics theory. Can. J. For. Res. **23:** 2499–2506.

Elfving, B., and A. Kiviste. 1997. Construction of site index equations for pinus sylvestris l. using permanent plot data in Sweden. For. Ecol. Man. 98: 125–134.

Eriksson H., U. Johansson, and A. Kiviste. 1997. A site-index model for pure and mixed tands of Betula pendula and Betula pubescens in Sweden. Scand. J. For. Res. 12: 149–156.

Kiviste, A. 1997. Eesti riigimetsa korguse, diameetri ja tagavara vanuseridade diferentsmudel 1984.–1993.a. metsakorralduse takseerkirjelduste andmeil. [Difference equations of stand height, diameter and volume depending on stand age and site factors for Estonian state forests (on the basis of 1984–1993 forest inventory data)]. - Eesti Pollumajandusulikooli Teadustoode kogumik. 189, 63–75. (Estonia).

Kiviste, A. 1998. Estimation of Estonian forest growth change in 1951–1994 on the basis of forest inventory data. - Climate change studies in Estonia. Tallinn, 1998. pp. 191–196.

MacKinney, A.L., F.X. Schumacher, and L.E. Chaiken. 1937. Construction of yield tables for non-normal loblolly pine stand. (USDA) J. Agric. Res. 54:531–545.

Monserud, R.A. 1984. Height growth and site-index curves for inland douglas-fir based on stem analysis data and forest habitat type. For. Sci. 30: 943–965.

Nigh, G.D. 1997. A Sitka spruce height-age model with improved extrapolation properties. For. Chron. 73: 363–369.

Ramanujan, A.S. 1985. Ramanujan's notebooks. Edited by Bruce C. Berndt. Ramanujan Aiyangar, Srinivasa, 1887–1920. New York: Springer-Verlag. c1985-v.: ports.; ISBN 0-387-96110-00; ISBN 3-540-96110-0.