One- and Two-point Principle Dynamic Site Equations Derived from Richards, Weibull, and other Exponential Functions

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ONE- AND TWO-POINT PRINCIPLE DYNAMIC SITE EQUATIONS DERIVED FROM RICHARDS, WEIBULL, AND OTHER EXPONENTIAL FUNCTIONS

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ABSTRACT. I present here a proper derivation of two-point principle dynamic equations from base models belonging to the class of the exponential models represented by Chapman-Richards, Weibull, Yang, and Bailey functions. The two-point principle dynamic equation site models offer extreme flexibility in modeling self-referencing dynamics where it might be desirable and is possible to use two points of observations (typically inventory measurements) as the reference points driving the model. Furthermore, using an example of the Chapman-Richards function, I present also a second derivation of such equations, which is not recommended for operational use in derivation of two-point principle models, but may be useful in stabilizing one-point principle equations for fitting purposes when the subject equation becomes unstable due to excessive propagation of round-off errors.

KEYWORDS: Base-age invariant, two-point principle, dynamic equations, site models, site index, polymorphism with variable asymptotes.

1 INTRODUCTION

The average height of dominant and co-dominant trees at a given age is a critical component of growth and yield models for southern pine plantations. This statistic is little affected by the stand densities that are normally encountered in managed plantations. Thus, site quality estimation procedures based on stand height data are the most commonly used techniques for evaluating site productivity (Clutter, *et.* al, 1983). Most of these height-based techniques for evaluation of site quality rely on the development of site-index-based height over age curves hereafter called site curves. Each site curve defines an expected height/age relationship referenced by the expected height at a specified base age.

In general, the most desirable data for the development of site index curves come from re-measured permanent sample plots or from stem analysis. Re-measurement or stem analysis data are also commonly combined by plots, resulting in an average height/age relationship for a given location. Both of these data sources provide a number of observed height/age pairs for a given location and allow flexibility in model forms and estimation techniques. The ultimate flexibility of these models has always been a prime target of research efforts. Over decades researchers have been exploring different approaches to increase model flexibility, varying from splicing pieces of curves from different equations (e.g., Borders et al. 1984), through developing increasingly more sophisticated equations (Cieszewski 2000, 2001, 2002, 2003, 2004), to using two reference points (Zeide 1978) instead of one in an approach labeled in this report hereafter as the two-point principle approach. Despite offering the ultimate flexibility the two-point-principle approach has never been widely adopted in forest management — possibly because it was introduced only in a table-defined environment without a readily available algebraic methodology allowing it to be structured in an equation framework, and likely because of its increased input requirements. Thus, developing an equation based framework for this approach might broaden its usefulness, assuming additional information availability, to at least some of the situations requiring additional flexibility.

I present here the algebraic framework for development of the two-point-principle models based on mathematical equations. Two-point principle dynamic site equations are similar to the traditional base-age-invariant (Bailey and Clutter 1974) dynamic site equations except they require two site indices instead of one. In appearance these equations are similar to two-point boundary solutions to differential equations (e.g., Schnute 1981) as well as to the duplex expected-value-parameter equations by Ratkowsky (1990). Yet, they can be used in a similar manner as the traditional site models but with two, instead of one, points of reference (e.g., two site indices with two different base ages), which could typically be inventory measurements at two different points in time.

2 BASE MODELS AND NOTATION

Throughout the rest of this report I will use the following notation defined in this section. The modeled phenomena of interest is a ***Y*** function of ***T***, that is: ***Y=f(T)***, in the relationship defined by the following base model:

$$Y = M\,T^{b} \tag{1}$$

where:

- ***M*** and ***b*** are the base model parameters;
- ***Y*** is a function of ***t***; and
- ***T*** is a transformation of the independent variable ***t***, such that:

  a. Model (1) can be equivalent to the Richards function (Richards 1959, Pienaar and Turnbull 1973) if:

  $$T = 1 - e^{(-a1\,t)} \tag{2}$$

  b. Model (1) can be equivalent to the Weibull based Yang function (Yang et al. 1978) if:

  $$T = 1 - e^{(-t^{a2})} \tag{3}$$

  c. Model (1) can be equivalent to the Bailey function (Bailey 1980) if:

  $$T = 1 - e^{(-a1\,t^{a2})} \tag{4}$$

d. Model (1) can be equivalent to the Cieszewski function[1] (Cieszewski and Bella 1991) if:

$$T = 1 - e^{\left(-a1\, t^{\left(\frac{1}{a2}\right)}\right)} \tag{5}$$

e. Model (1) can be equivalent to any other model form that makes sense within this general model scheme.

Furthermore, the subscript "$_0$" will indicate a reference point introduced during derivation of one-point principle dynamic equations, and the subscripts "$_1$" and "$_2$" will indicate the reference points introduced in a derivation of two-point principle dynamic site equations. Similarly $\boldsymbol{Y_0}$ will be a function of $\boldsymbol{T_0}$, which will signify the use of $\boldsymbol{t_0}$.

The linear model multiplier $\boldsymbol{M}$ will be at times redefined arbitrarily as " $\boldsymbol{e^m}$ ", where $\boldsymbol{m}$ will be the re-parameterization parameter, so that the function will take the form of:

$$Y = e^m\, T^b \tag{6}$$

Finally, the symbols $\boldsymbol{LY}$ and $\boldsymbol{LT}$ will denote the natural log of $\boldsymbol{Y}$ and $\boldsymbol{T}$, that is $\boldsymbol{Ln(Y)}$ and $\boldsymbol{Ln(T)}$, so that the log-transform of model (1) will be:

$$LY = m + b\, LT \tag{7}$$

3 ONE-POINT PRINCIPLE DYNAMIC SITE EQUATIONS BASED ON THE ALGEBRAIC DIFFERENCE APPROACH

If $\boldsymbol{Y_0}$ defines the reference point at time $\boldsymbol{T_0}$, the Algebraic Difference Approach (ADA) (Bailey and Clutter 1974) formulations based on solving model (1) for $\boldsymbol{M}$ is:

$$Y_M = Y_0 \left(\frac{T}{T_0}\right)^b \tag{8}$$

for $\boldsymbol{b}$ is:

$$Y_b = M\, T^{\left(\frac{\ln\left(\frac{Y_0}{M}\right)}{\ln(T_0)}\right)} \tag{9}$$

and assuming, for example, $\boldsymbol{T}$ defined by the Chapman-Richards special case of the Bailey function (4):

[1] which is a modified Bailey function believed to be more stable with the parameters *a2* and *b* in inverse to each other.

$$T = 1 - e^{(-a1\, t^{a2})} \tag{10}$$

the ADA solution based on the parameter ***a*** is:

$$Y_a = M\left(1 - \left(-\left(\frac{Y_0}{M}\right)^{\left(\frac{1}{b}\right)} + 1\right)^{\left(t_0^{(-a2)}\, t^{a2}\right)}\right)^b \tag{11}$$

while for the assumption of the Yang special case of the modified Bailey function (3) the solution is:

$$Y_a = M\left(1 - e^{\left(-a1\left(-\frac{\ln\left(-\left(\frac{Y_0}{M}\right)^{\left(\frac{1}{b}\right)} + 1\right)}{\ln(e)\, a1}\right)^{\left(\frac{\ln(t)}{\ln(t_0)}\right)}\right)}\right)^b \tag{12}$$

4 TWO-POINT PRINCIPLE DYNAMIC SITE EQUATIONS

The proper derivation of the two-point principle model is based on the following procedure. First, equation (1) can be solved for $\boldsymbol{M}$ as a function of $\boldsymbol{Y_1}$ of $\boldsymbol{T_1}$, that is $\boldsymbol{Y_1=f(T_1)}$, which is the first reference point signifying also the use of $\boldsymbol{t_1}$ (in $\boldsymbol{T_1}$):

$$M = \frac{Y_1}{T_1^{\,b}} \tag{13}$$

which substituted in the base model (1) gives:

$$Y = Y_1 \left(\frac{T}{T_1}\right)^b \tag{14}$$

Then the second point solution can be obtained by solving equation (14) for the parameter $\boldsymbol{b}$ as a function $\boldsymbol{Y_2}$ of $\boldsymbol{T_2}$, that is $\boldsymbol{Y_2=f(T_2)}$, which is the second reference point signifying the presence of $\boldsymbol{t_2}$ (in $\boldsymbol{T_2}$):

$$b = \frac{\ln\left(\frac{Y_2}{Y_1}\right)}{\ln\left(\frac{T_2}{T_1}\right)} \tag{15}$$

Substituting this solution in equation (1) results in the two-point principle solution equation having only one parameter ***a*** in ***T***:

$$Y = Y_1 \left(\frac{T}{T_1}\right)^{\left(\frac{\ln\left(\frac{Y_2}{Y_1}\right)}{\ln\left(\frac{T_2}{T_1}\right)}\right)} \tag{16}$$

which is also equivalent to:

$$Y = Y_1 \left(\frac{Y_2}{Y_1}\right)^{\left(\frac{\ln\left(\frac{T}{T_1}\right)}{\ln\left(\frac{T_2}{T_1}\right)}\right)} \tag{17}$$

A two-point solution can also be derived from one of the existing GADA models; however, this is not recommended due to relatively easy confusion resulting in possible errors and in inferior model form. For example, let's consider the following GADA model derived by Cieszewski (2004):

$$Y = e^{X_1}\, T^{\left(b1 + \frac{b2}{X_1 + b3}\right)} \tag{18}$$

where:

- ***b1***, ***b2***, and ***b3*** are the model parameters,
- $X_1 = -b3 - .5\, v + .5 \sqrt{-4\, LT_1\, b2 + v^2}$ (19)
- $v = LT_1\, b1 - LY_1 - b3$
- and the symbols LY_1 and LT_1 denote the natural log of Y_1 and T_1, that is $Ln(Y_1)$ and $Ln(T_1)$, T_1 is a function of t_1 and the subscript "$_1$" signifies the first reference point.

I substitute $M=exp(X_1)$, which will be here solved for, as in equation (13), while introducing the second reference point:

$$Y = M\, T^{\left(b1 + \frac{b2}{X_1 + b3}\right)} \tag{20}$$

which subsequently can be isolated from the main equation (18) using an arbitrary point $\{T_2, Y_2\}$ denoted with subscript "$_2$" to distinguish it from the other point of reference introduced by GADA during derivation of equation (18):

$$M = \frac{Y_2}{T_2^{\left(b1 + \frac{b2}{X_1 + b3}\right)}} \tag{21}$$

This operation is somewhat similar to the model conditioning used by Burkhart and Tennant (1977). The substituting of ***M*** in equation (18) with the right hand side solution of equation (21) produces the following two-point principle model equivalent to models (16) and (17):

$$Y = Y_2 \left(\frac{T}{T_2}\right)^{\left(b1 + \frac{b2}{X_1 + b3}\right)} \tag{22}$$

where:

- ***b1***, ***b2***, and ***b3*** are the model parameters,
- $X_1 = -b3 - .5\,v + .5\,\sqrt{-4\,LT_1\,b2 + v^2}$
- $v = LT_1\,b1 - LY_1 - b3$
- the symbols LY_1 and LT_1 denote the natural log of Y_1 and T_1, that is $Ln(Y_1)$ and $Ln(T_1)$, T_1 is a function of t_1 and the subscript "$_1$ " signifies the first reference point.
- and the symbols LY_2 and LT_2 denote the natural log of Y_2 and T_2, that is $Ln(Y_2)$ and $Ln(T_2)$, T_2 is a function of t_2 and the subscript "$_2$ " signifies the second reference point that is independent from the first reference point, but in this equation can be equal to it.

Model (22) is inferior to model (16) and (17) because it has unnecessarily four times as many parameters without any additional flexibility in curve shapes or represented curve patterns, even though it has a minor advantage over the later models because it can be used with just one reference point when $T_1=T_2$ and $Y_1=Y_2$. Since one- and two-point principle models are not cross compatible or exchangeable, in general the derivation of such dual-purpose models is not advisable in operational use. Analogically, models similar to model (22) should not be taken for granted or misrepresented as solely one-point principle models by merely rewriting the subscripts "$_2$ " to "$_1$ " or changing both of the subscripts to "$_0$ " without explanation of the derivation process. Such practices may easily lead to development of ill-conditioned and internally inconsistent models, such as all those identified in Bailey and Cieszewski (2000).

Model (22) is more complicated and it will compute more slowly than the equivalent model (18). However, the practitioners may find that despite more complex appearance and slower computing, model (22) may be superior to model (18) in fitting as a one-point principle model due to its potentially higher robustness in the presence of round-off errors and parameter estimation instability. This might be due to the fact that the ratio of (T/T_2) generally computes more stable values than T alone, and the values of Y_2 are generally more stable than values of $exp(X_1)$ (Cieszewski and Bella 1991).

5 EXAMPLES OF EQUATION IMPLEMENTATION

To implement any of the presented equations, substitute all the relevant sub-equations until the model computes "***Y*** " directly from the following symbols, which have to be assigned actual numerical values (i.e.,: concrete numbers) in any computer code used to compute any of the given equations:

- the values of the independent variable " ***t*** ",
- the values of the one-point-principle reference point $\{t_0;Y_0\}$, or the values of the two two-point-principle reference points $\{t_1;Y_1\}$ **and** $\{t_2;Y_2\}$, and
- the values of the parameters ***a1***…, ***b1***…, and ***m1***….

For example, to use equation (17), representing a two-point-principle Richards based site model, the substitutions should replace T, T_1, and T_2 with the right-hand-side of equation (2) using consecutively: t, t_1, and t_2 as their independent variables (instead of t), which in FORTRAN could be programmed as follows:

```
T  = 1-e**(-a1*t)
T1 = 1-e**(-a1*t1)
T2 = 1-e**(-a1*t2)
Y  = Y1*(Y2/Y1)**(alog(T/T1)/alog(T2/T1))
```

Given the values of the underlined symbols**,** and providing that neither **Y1** and **Y2,** nor **t1** and **t2,** are equal, the above code will compute the value of **Y** according to the relationship defined by equation (17).

Where logarithms are involved the substitution functions need to account for the computation of the logs. For example, to compute the one-point-principle equation (18) for the Richards case the code can be:

```
T   = 1-e**(-a1*t)
LT1 = alog(1-e**(-a1*t1))
v   = b1*LT1-alog(Y1)-b3
X1  = -b3-0.5*(v+sqrt(-4*LT1*b2+v**2))
Y   = e**X1*T**(b1+b2/(X1+b3))
```

But, to compute the two-point-principle equation (22) using the values of the parameter **a1, b1, b2,** and **b3**, and the values of the variables: **t, t1, t2, Y1**, and **Y2,** the code should be:

```
T   = 1-e**(-a1*t)
LT1 = alog(1-e**(-a1*t1))
T2  = 1-e**(-a1*t2)
v   = b1*LT1-alog(Y1)-b3
X1  = -b3-0.5*(v+sqrt(-4*LT1*b2+v**2))
Y   = Y2*(T/T2)**(b1+b2/(X1+b3))
```

However, since in this case it is permissible to have **Y1**=**Y2,** and **t1**=**t2,** the following code can also be used for a one-point-principle computation equivalent to equation (18) if, say, equation (18) appears to be numerically unstable:

```
t2  = t1
```

```
Y2  = Y1
T   = 1-e**(-a1*t)
LT1 = alog(1-e**(-a1*t1))
T2  = 1-e**(-a1*t2)
v   = b1*LT1-alog(Y1)-b3
X1  = -b3-0.5*(v+sqrt(-4*LT1*b2+v**2))
Y   = Y2*(T/T2)**(b1+b2/(X1+b3))
```

Given the values of the parameter **a1, b1, b2,** and **b3**, and the variables: **t, t1,** and **Y1,** the above code will compute the value of **Y** according to the relationship defined by equation (18) and the restricted case of equation (22) with only one reference point (that is a single site index).

6 Summary

I presented here derivation of two-point principle models derived from the Richards, Weibull, Yang, Bailey, and Cieszewski's functions. The presented models can be used for modeling families of growth series using two inventory measurements as driving variables. I demonstrated two approaches to the derivation of two-point principle models, but only the first approach is proper for this purpose and can be recommended for operational use. Ironically the other presented approach to two point-principle model derivations may be recommended for stabilizing one-point principle models in the event that the model is numerically unstable during the model fitting. In short, the first approach results in superior two-point principle models that cannot be used as one-point principle models, while the other approach results in inferior two-point principle models that can be used as one-point principle models that under certain circumstances may have improved numerical stability. Readers interested in looking for other examples of equations applicable to two-point principle modeling may find various materials in Schnute (1981), Ratkowsky (1990) and other literature on two-point boundary solutions to differential equations. Those interested in the issues of numerical stability of various equation parameterizations, especially of the exponential class of models, such as the presented here equations, may refer to Cieszewski and Bella (1991).

7 Disclaimer

My past work with the two-point principle systems did not indicate that there was much to gain from having a second reference point in models. Accordingly, I have mixed feelings and reservations against recommending this system or building up any strong expectations of it. If my past experience with these systems is of any general applicability, the modelers will find with them that first there is so much information coming from the first point that the second does not have much to contribute to, and second, that the noise from natural variation has a detrimental impact on these systems because they are too sensitive to error propagation problems.

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APPENDIX A: FORTRAN CODE FOR THE PRESENTED EQUATIONS

Code by equation numbers:

```
(1)   Y = M*T**b
(2)   T = 1-e**(-a1*t)
(3)   T = 1-e**(-t**a2)
(4)   T = 1-e**(-a1*t**a2)
(5)   T = 1-e**(-a1*t**(1/a2))
(6)   Y = e**m*T**b
(7)   LY = m+b*LT
(8)   YM = Y0*(T/T0)**b
(9)   Yb = M*T**(alog(Y0/M)/alog(T0))
(10)  T = 1-e**(-a1*t**a2)
(11)  Ya= M*(1-(-(Y0/M)**(1/b)+1)**(t0**(-a2)*t**a2))**b
(12)  Ya= M*(1-e**(-a1*(-alog(-
    (Y0/M)**(1/b)+1)/alog(e)/a1)**(alog(t)
      a. #/alog(t0))))**b
(13)  M = Y1/T1**b
(14)  Y = Y1/T1**b*T**b
(15)  b = alog(Y2/Y1)/alog(T2/T1)
(16)  Y = Y1*(T/T1)**(alog(Y2/Y1)/alog(T2/T1))
(17)  Y = Y1*(Y2/Y1)**(alog(T/T1)/alog(T2/T1))
(18)  Y = e**X1*T**(b1+b2/(X1+b3))
(19)  X1 = -b3-0.5*(v+sqrt(-4*LT1*b2+v**2)), where v=LT1*b1-
    LY1-b3
(20)  Y = M*T**(b1+b2/(X1+b3))
(21)  M = Y2/T2**(b1+b2/(X1+b3))
(22)  Y = Y2*(T/T2)**(b1+b2/(X1+b3))
```